Archimedes, the Sand Reckoner

Ilan Vardi*

Contents

*IHES, 35, route de Chartres, 91440 Bures sur Yvette, France, ilan@ihes.fr

1 Translation

I. There are some, king Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognizing that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe. Now you are aware that 'universe' is the name given by most astronomers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account, as you have heard from astronomers. But Aristarchus of Samos brought out a book consisting of some hypotheses, in which the premisses lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. Now it is easy to see that this is impossible. For, since the centre of the sphere has no magnitude, we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must however take Aristarchus to mean this: Since we conceive the earth to be, as it were, the centre of the universe, the ratio which the earth bears to what we describe as the "universe" is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars. For he adapts the proofs of his results to a hypothesis of this kind, and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the "universe."

I say then that, even if a sphere were made up of sand as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that, of the numbers named in the *Principles,* some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made.

First, the perimeter of the earth is three hundred myriad stadia and no greater, though some have tried to show, as you know, that this length is thirty myriad stadia. But I, surpassing this number and setting the size of the earth as being ten times that evaluated by my predecessors, suppose that its perimeter is three hundred myriad stadia and not greater.

2. Second, that the diameter of the earth is greater than the diameter of the moon and that the diameter of the sun is greater than the diameter of the earth. My hypothesis is in agreement with most earlier astronomers.

3. Third hypothesis: the diameter of the sun is thirty times larger than that of the moon and not greater, even though among earlier astronomers Eudoxus tried to show it as nine times larger and Pheidias,

my father, as twelve times larger, while Aristarchus tried to show that the diameter of the sun lies between a length of eighteen moon diameters and a length of twenty four moon diameters; but I, surpassing this number as well, suppose, so that my proposition may be established without dispute, that the diameter of the sun is equal to thirty moon diameters, and not more.

4. Finally, we state that the diameter of the sun is greater than the side of the polygon of one thousand sides inscribed in the great circle of the universe. I make this hypothesis because Aristarchus found that the sun appears as the seven hundred and twentieth part of the circle of the zodiac. While examining this question I have, for my part tried in the following manner, to show with the aid of instruments, the angle subtended by the sun, having its vertex at the eye. Clearly, the exact evaluation of this angle is not easy since neither vision, hands, nor the instruments required to measure this angle are reliable enough to measure it precisely. But this does not seem to me to be the place to discuss this question at length, especially because observations of this type have often been reported. For the purposes of my proposition, it suffices to find an angle that is not greater than the angle subtended at the sun with vertex at the eye and to then find another angle which is not less than the angle subtended by the sun with vertex at the eye. A long ruler having been placed on a vertical stand placed in the direction of where the rising sun could be seen, and a little cylinder was put vertically on the ruler immediately after sunrise. Then, the sun being at the horizon, and could be looked at directly, the ruler was oriented towards the sun and the eye at the extremity of the ruler. The cylinder being placed between the sun and the eye, occludes the sun. The cylinder is then moved further away from the eye and as soon as a small piece of the sun begins to show itself from each side of the cylinder, it is fixed. If the eye were really to see from one point, tangents to the cylinder produced from the end of the ruler where the eye was placed would make an angle less than the angle subtended by the sun with vertex at the eye. But since the eyes do not see from a unique point, but from a certain size, one takes a certain size, of round shape, not smaller than the eye and one places it at the extremity of the ruler where the eye had been placed. If one produces tangents to this size and to the cylinder, the angle between these lines is smaller than the angle subtended by the sun with vertex at the eye. And here is the way one finds the size not smaller to the eye: one takes two small thin cylinders of the same width, one white, the other not, and one places them in front of the eye, the white one at some distance, and the other one which is not white as close to the eye as possible without touching the face. In this way, if the small cylinders chosen are smaller than the eye, the cylinder neighboring the eye is encompassed in the visual field and the eye sees the white cylinder. If the cylinders are much smaller, the white one is completely seen. If they are not much smaller, one sees parts of the white one and parts of the one neighboring the eye. But if one choose cylinders of appropriate width one of them occludes the other without covering a larger space. It is therefore clear that the width of cylinders producing this effect is not smaller than the dimensions of the eye. As for the angle not smaller than the angle subtending the sun with vertex at the eye, it was taken as follows: The cylinder being placed on the ruler at a distance which blocks all of the sun, if one produces from the end of the ruler where the eye is placed tangent lines to the cylinder, the angle made by these lines is not smaller than the angle subtended by the sun with vertex at the eye. A right angle being measured by the angles taken in this way, the angle placed at the point is found to be the one hundred and sixty fourth part of a right angle, while the smallest angle is found to be greater to the two hundredth part of a right angle. It is therefore clear that the angle subtended by the sun with vertex at the eye is also smaller than the one hundred and sixty fourth part of a right angle, and greater than the two hundredth part of a right angle.

With these measurements completed, one shows that the diameter of the sun is greater than the side of the polygon with one thousand sides inscribed in the great circle of the universe. Let us imagine then a plane passing through the centre of the sun, the centre of the earth and the eye at the instant when the sun finds itself a little above the horizon; that this plane cuts the universe at the circle $AB\Gamma$, the earth at the circle ΔEZ, and the sun at the circle ΣH. Let Θ be the centre of the earth, K the centre of the sun, and let Δ be the eye; we produce from Δ the tangents $\Delta\Lambda$, $\Delta\Xi$ to the circle ΣH with contact points N and T, and from Θ the tangents ΘM and ΘO with contact points X and P. Let A and B be the points of intersection of the circle $AB\Gamma$ and the lines ΘM and ΘO. Thus ΘK is greater than ΔK from the hypothesis that the sun finds itself above the horizon. If follows that the angle contained between $\Delta\Lambda$ and $\Delta\Xi$ is greater than the angle contained between ΘM and ΘO. But the angle contained between $\Delta\Lambda$ and $\Delta\Xi$ is greater than the two hundredth part of a right angle since it is equal to the angle subtended by the sun with vertex at the eye; and consequently, the angle contained between ΘM and ΘO is less than the one hundred and sixty fourth part of a right angle and the segment of the line AB is less than the chord of the circular sector which is the six hundred and sixty fifth part of the circle $AB\Gamma$. But the perimeter of the polygon in question has with the radius of the circle $AB\Gamma$ a ratio less than fourty four to seven because the ratio of the perimeter of every polygon inscribed in a circle to the radius of the circle is less than the ratio fourty four to seven. You know, in fact, that I have shown that in every circle the perimeter is greater, by a quantity smaller than the seventh, than triple the diameter and that the perimeter of the inscribed polygon is smaller than this circumference. The ratio of BA to ΘK is thus less than the ratio of eleven to one thousand one hundred and fourty eight. It follows that BA is smaller than a hundredth ΘK. But the diameter of the circle ΣH is equal to BA since half of ΣH, the segment ΦA, is equal to KP. The segments ΘK and ΘA are in fact equal and from their endpoints perpendiculars are produced of equal angle. It is thus clear that the diameter of the circle ΣH is less than the hundredth part of ΘK. Moreover, the diameter $E\Theta\Upsilon$ is less than the diameter of the circle ΣH since the circle ΔEZ is less than the circle ΣH. If follows that the sum of $\Theta\Upsilon$ and $K\Sigma$ is less than the hundredth part of ΘK so that the ratio of ΘK to ΥK is less than the ratio of one hundred to ninety nine. And as long as $\Sigma\Upsilon$ is less than ΔT, the ratio of ΘP to ΔT is less than the ratio of one hundred to ninety nine. But since in the right triangles ΘKP and ΔKT the sides KP and KT are equal and the sides ΘP and ΔT unequal, ΘP being larger, the ratio of the angle contained between the sides ΔT and ΔK to the angle contained between the ΘP and ΘK is greater than the ratio of ΘK to ΔK, but less than the ratio of ΘP to ΔT. For if in two right triangles two of the sides containing the right angle are equal and the two others unequal, then the larger angle opposite the unequal sides has to the smaller of these angles a ratio greater than the ratio of the greater hypotenuse to the smaller, but smaller than the ratio of the greater side to the right angle to the smaller. As a consequence, the ratio of the angle contained between $\Delta\Lambda$ and $\Delta\Xi$ to the angle contained between ΘO and ΘM is less than the ratio of ΘP to ΔT which is itself less than the ratio of one hundred to ninety nine. It follows that the ratio of the angle contained between $\Delta\Lambda$ and $\Delta\Xi$ is greater than the two hundredth part of a right angle, the angle contained between ΘM and ΘO is greater than ninety nine twenty thousandths of a right angle; and as a consequence, this angle is greater than one two hundred and third of a right angle. The segment BA is thus greater than the chord of the sector which is a eight hundred and twelfth part of the circle $AB\Gamma$. But it is to the line segment AB that the diameter of the circle is equal to. It is therefore clear that the diameter of the circle is greater than the side of the polygon of one thousand sides.

II. These relations being given, one can also show that the diameter of the universe is less than a line equal to a myriad diameters of the earth and that, moreover, the diameter of the universe is less than a line equal to one hundred myriad myriad stadia. As soon as one has accepted the fact that the diameter of the sun is not greater than thirty moon diameters and that the diameter of the earth is greater than the diameter of the moon, it is clear that the diameter of the sun is less than thirty diameters of the earth. As we have also shown that the diameter of the sun is greater than the side of the polygon of one thousand sides inscribed in the great circle of the universe, it is clear that the perimeter of the indicated polygon of one thousand sides is less than one thousand diameters of the sun. But the diameter of the sun is less than thirty earth diameters so it follows that the perimeter of the polygon of one thousand sides is less than thirty thousand earth diameters. Given that the perimeter of the polygon of one thousand sides is less than thirty thousand earth diameters and greater than three diameters of the universe–we have shown in fact that in every circle the diameter is less than one third the perimeter of any regular polygon inscribed in the circle for which the number of sides is greater than that of the hexagon–the diameter of the universe is less than a myriad earth diameters. One has thus shown that the diameter of the universe is less than a myriad earth diameters; that the diameter of the universe is less than one hundred myriad myriad stadia, which comes out of the following argument; since, in fact, we have supposed that the perimeter of the earth is not greater than three hundred myriad stadia and that the perimeter of the earth is greater than triple the diameter because in every circle the circumference is greater than triple the diameter, it is clear that the diameter of the earth is less than one hundred myriad stadia. Given that the diameter of the universe is less than a myriad earth diameters it is clear that the diameter of the world is less than one hundred myriad myriad stadia. These are my hypotheses regarding sizes and distances. Here now is what I assume about the subject of sand: if one has a quantity of sand whose volume does not exceed that of a poppy–seed, the number of these grains of sand will not exceed a myriad and the diameter of the grains will not be less than a fourtieth of a finger–breadth. I make these hypotheses following these observations: poppy seeds having been placed on a polished ruler in a straight line in such a way that each touches the next, twenty five seeds occupied a space greater than one finger–breadth. I will suppose that the diameter of the grains is smaller and to be about a fourtieth of a finger–breadth for the purpose of removing any possibility of criticizing the proof of my proposition

III. These are thus my hypotheses; but I think it useful to explain myself about the naming of numbers so that those readers, not having been able to get hold of my book addressed to Zeuxippus, may not be thrown off by the absence in this book of any indication of the subject of this nomenclature. It so happens that tradition has given to us the name of numbers up to a myriad and we distinguish enough numbers surpassing a myriad by enumerating the number of myriads until a myriad myriad. We will therefore call first numbers those which, according to this nomenclature, go up to a myriad myriad. We will call units of second numbers the myriad myriad of first numbers and we will count among second number units and, starting with units, tens, hundreds, thousands, myriads, until a myriad myriad. We will call once again call third numbers a myriad myriad of second numbers and we will count among third numbers, starting with units, tens, hundreds, thousands, myriads, until the myriad myriad. In the same way we will call units of fourth numbers a myriad myriad of third numbers, units of fifth numbers a myriad myriad of fourth numbers, and continuing in this way the numbers will be distinguishable until the myriad myriad of of myriad myriad numbers.

Numbers named in this way could certainly suffice but it is possible to go still further. Let us in fact call numbers of the first period the numbers given up to this point and units of first numbers of the second period the last number of the first period. Furthermore, call the unit of second numbers of the second period the myriad myriad of first numbers of the second period. In the same way, the last of these numbers will be called the unit of third numbers of the second period, and continuing in this way, progressing through the numbers of the second period will have their names up to the myriad myriadth of myriad myriad numbers. The last number of the second period will be in turn called the unit of the first numbers of the third period, and so forth until a myriad myriad units of myriad myriadth numbers of the myriad myriadth period.

These numbers having been named, if numbers are ordered by size starting from unity and if the number closest to unity is the tens, the first eight of these including the unity will belong to the numbers called first numbers, the following eight numbers called second, and the others in the same way by the distance of their octad of numbers to the first octad of numbers. The eighth number of the first octad is thus one thousand myriads and the first number of the second octad, since it multiplies by ten the number preceding it, will be a myriad myriad and this number is the unit of the second numbers. The eight number of the second octad is one thousand myriad of second numbers. The first number of the third octad will once again be, as it multiplies by ten the preceding number, a myriad myriad of second numbers, the unity of the third numbers. It is clear that the same will hold as indicated for any octad.

It is useful to know what follows. If numbers are in proportion starting from unity and some which are in the same proportion are multiplied to each other, then the product will be increased from the larger of the factors by as many numbers as the smaller number is far in proportion to unity and it will be increased from unity by the sum minus one of the distance of the numbers away from unity. In fact, let $A, B, \Gamma, \Delta, E, Z, H, \Theta, I, K, \Lambda$ be in proportion starting from unity, and let A be unity. Multiply Δ by Θ and let X be the product. Let us take in the proportion Λ whose distance to Θ holds as many numbers as the distance from Δ to unity. It must be shown that X equals Λ. If, among the numbers in proportion, the distance from Δ to A counts as many numbers as that from Λ to Θ, the ratio of Δ to A equals the ratio of Λ to Θ. But Δ is the product of Δ by A from which it follows that Λ is the product of Δ by Θ, so Λ is equal to X. It is therefore clear that the product is in the proportion and that its distance to the largest factor counts as many numbers as the distance of the smaller factor to unity. But it is also clear that this product is increased, from unity, by the sum minus one, of the distances of the numbers Δ and Θ to unity; for $A, B, \Gamma, \Delta, E, Z, H, \Theta$ are the numbers by which Θ is increased from unity, and I, K, Λ are, up to a number, those by which Δ is increased from unity; by adding Θ one gets the sum of the distances.

IV. The preceding being in part assummed and in part proved, I will now prove my proposition. As we have assumed that the diameter of a poppy–seed is not smaller than a fourtieth of a finger–breadth, it is clear that the volume of the sphere having diameter one finger–breadth does not exceed that of sixty four thousand poppy–seeds; for this number indicates how many times it is the multiple of the sphere having as diameter one fourtieth of a finger–breadth; it has in fact been shown that spheres are related to each other as the cubes of their diameters. As we have also assumed that the number of grains of sand contained in one poppy–seed does not exceed a myriad, it is clear that, if the sphere having diameter one finger–breadth were filled with sand, the number of grains would not exceed sixty four thousand myriads. But this number represents six units of second numbers increased by four thousand myriad of first numbers, and is thus less than ten units of second numbers. The sphere with diameter one hundred finger–breadths is equivalent to one hundred myriad spheres of diameter one finger–breath, since spheres are related to each other as the cubes of their diameters. If one now had a sphere filled with sand of the size of the sphere of diameter one hundred finger–breadths, it is clear that the number of grains of sand would be less than the product of ten myriad second numbers and one hundred myriads. But since ten units of second numbers make up the tenth number starting from unity in the proportional sequence of multiple ten, and the one hundred myriads of the seventh number starting from unity in the same proportional sequence, it is clear that the number obtained will be the sixteenth starting from unity in the same proportional sequence. For we have shown that the distance of this product to unity is equal to the sum of, minus one, of the distance from unity of its two factors. From these sixteen numbers the first eight are among, with unity, the numbers called first numbers, the following eight are part of the second numbers, and the last of these is one thousand myriad second numbers. It is now evident that the number of grains of sand whose volume is equal to one hundred finger–breadths is less than one thousand myriad second numbers. Similarly, the volume of the sphere of diameter one myriad finger–breadths is one hundred myriad times the volume of the sphere of diameter one hundred finger–breadths. If one now had a sphere, filled with sand, of the size of the sphere with diameter a myriad finger breadths, it is clear that the number of grains of sand would be less than the product of one thousand myriads of second numbers and one hundred myriads. But since one thousand myriad second numbers are the sixteenth number starting from unity in the proportional sequence and that one hundred myriad are the seventh number starting from unity in the same proportional sequence, it is clear that the product will be the twenty second number starting from unity in the same proportional sequence. Of these twenty two numbers, the first eight, with unity, are among the numbers called first numbers, the following eight are among the numbers called second, and the six remaining numbers are called third numbers, the last of which being ten myriad third numbers. It is then clear that the number of grains of sand whose volume is equal to a sphere of diameter of a myriad finger–breadths is less than ten myriads of third numbers. And since the sphere with diameter one stade is smaller than the sphere with diameter a myriad finger–breadths, it is also clear that the number of grains of sand contained in a volume equal to a sphere with diameter one stade is less than ten myriad third numbers. Similarly, the volume of a sphere of diameter one hundred stadia is one hundred myriad times the volume of a sphere of diameter one stade. If one now had a sphere, filled with sand, of the size of the sphere with diameter one hundred stadia, it is evident that the number of grains of sand would be less than the product of ten myriad third numbers with one hundred myriad. And since ten myriad third numbers are the twenty second numbers, starting from unity, in the proportional sequence, and that one hundred myriad are the seventh number starting from unity in the same proportional sequence, it is clear that the product will be the twenty eighth number starting from unity in the proportional sequence. Of these twenty eight numbers, the first eight, with unity, are part of the numbers called first numbers, the following eight are second numbers, the following eight are third numbers, and the four remaining are called fourth, the last being one thousand units of fourth numbers. It is then evident that the number of grains of sand whose volume equals that of a sphere of diameter a hundred stadia is less than one thousand units of fourth numbers. Similarly, the volume of a sphere of diameter a myriad stadia is one hundred myriad times the volume of a sphere having diameter one hundred stadia. If one then had a sphere, filled with sand, of the size of a sphere of diameter a myriad stadia, it is clear that the number of grains of sand would be less than the product of one thousand units of fourth numbers with one hundred myriad. Just as one thousand units of fourth numbers represent the twenty eighth number, starting from unity, in the proportional sequence, and one hundred myriad the seventh number in the proportional sequence, starting from unity, of the same proportional sequence, it is clear that their product will be, in the same proportional sequence, with unity, the thirty fourth number starting form unity. But of these thirty four numbers, the first eight, with unity, are among those numbers called first numbers, the following eight among second numbers, the following eight among third numbers, the following eight among fourth numbers, and the two remaining among fifth numbers, the last of these being ten units of fifth numbers. It is thus clear that the number of grains of sand whose volume is equal to that of a sphere having diameter a myriad stadia will be smaller than ten units of fifth numbers. And similarly, the volume of a sphere of diameter one hundred myriad stadia is one hundred myriad times the volume of a sphere of diameter a myriad stadia. If one had then had a sphere, filled with sand, of the size of the sphere with diameter one hundred myriad stadia, it is clear that the number of grains of sand would be smaller than the product of ten units of fifth numbers and one hundred myriads. As the ten units of fifth numbers represent the thirty fourth number starting from unity in the proportional sequence, and one hundred myriads the seventh number starting from unity in the same proportional sequence, it is clear that the product will be, in the same proportional sequence, the fourtieth number starting from unity. But of these fourty numbers, the first eight, with unity, are among the numbers called first numbers, the eight following are second numbers, the eight following are third numbers, the eight following are fourth numbers, the eight following are fifth numbers, the last of these being one thousand myriad fifth numbers. It is therefore clear that the number of grains of sand whose volume is equal to that of a sphere of diameter one hundred myriad stadia is less than one thousand myriad fifth numbers. But the volume of a sphere of diameter a myriad myriad stadia is one hundred myriad times the sphere of diameter one hundred myriad stadia. Thus, if one had a sphere, filled with sand, of the size of a sphere of diameter a myriad myriad stadia, it is clear that the number of grains of sand would be less than the product of one thousand myriad fifth numbers by one hundred myriads. However, since one thousand myriad fifth numbers represent the fourtieth number, starting from unity, of the proportional sequence, and one hundred myriad the seventh number starting from unity in the same proportional sequence, it is clear that the product will be the fourty sixth number starting from unity. Of these fourty six numbers, the first eight, with unity, are part of the numbers called first numbers, the eight following second numbers, the eight following third numbers, the eight following fourth numbers, the eight following fifth numbers, and the six left over are numbers called sixth, the last among being ten myriads of sixth numbers. It is thus clear that the number of grains of sand whose volume is equal to a sphere of diameter a myriad myriad stadia is smaller than ten myriad sixth numbers. But the volume of a sphere of diameter one hundred myriad myriad stadia is one hundred myriad times the multiple of a sphere of diameter a myriad myriad stadia. Thus, if one had a sphere, filled with sand, of the size of a sphere of diameter one hundred myriad myriad stadia, it is clear that the number of grains of sand would be smaller than the product of ten myriad sixth numbers by one hundred myriad. But, since ten myriad sixth numbers represent the fourty sixth number, starting from unity, in the proportional sequence, and one hundred myriad the seventh number starting from unity in the same proportional sequence, it is clear that the product will be the fifty second number starting in the same proportional sequence. But of these fifty two numbers, the first fourty eight, with unity, belong to numbers called first numbers, second numbers, third, fourth, fifth, and sixth, and the the four remaining are among numbers called seventh numbers, the last of them being one thousand units of seventh numbers. It is thus clear that the number of grains of sand in a volume equal to a sphere whose volume is equal to that of a sphere of diameter one hundred myriad

myriad stadia is smaller than one thousand units of seventh numbers.

As we shown that the diameter of the universe is less than one hundred myriad myriad stadia, it is clear that the number of grains of sand filling a volume equal to that of the universe is itself less than one thousand units of seventh numbers. We have thus shown that the number of grains of sand filling a volume equal to that of the universe, as the majority of astronomers understand it, is one thousand units of seventh numbers; we will now show that even the number of grains of sand filling a volume equal to the sphere as large as Artistarchus proposed for the fixed stars, is smaller than one thousand myriad eighth numbers. As we have assumed, in fact, that the ratio of the earth to what we commonly call the universe is equal to the ratio of this universe to the sphere of fixed stars as proposed by Aristarchus, the two spheres have the same ratio to each other. But it has been shown that that the diameter of the universe is less than a length a myriad times the multiple of the diameter of the earth. It is thus clear that the diameter of the sphere of fixed stars is itself smaller to a length a myriad times the diameter of the universe. But since the sphere have the ratio among themselves of their diameters, it is clear that the sphere of fixed stars, as Aristarchus proposes, is less than a volume a myriad myriad myriad times a multiple the volume of the universe. But we have shown that the number of grains of sand filling a volume equal to that of the world is less than a thousand units of seventh numbers; it is therefore evident that that if a sphere, as large as Aristarchus supposes that of the fixed stars to be, were to be filled with sand, the number of grains of sand would be less than the product of one thousand units [of seventh numbers] by a myriad myriad myriad. And since one thousand units of seventh numbers represent the fifty second number in the reciprocal sequence starting from unity, and a myriad myriad myriads the thirteenth number starting from unity in the same proportional sequence, it is clear that the product will be the sixty fourth number starting from unity in the same proportional sequence; but this number is the eighth of the eight numbers, which is one thousand myriads of eight numbers.

It is therefore obvious that the number of grains of sand filling a sphere of the size that Aristarchus lends to the sphere of fixed stars is less than one thousand myriad eighth numbers.

I conceive, King Gelon, that among men who do not have experience of mathematics, such a thing might appear incredible. On the other hand, those who know of such matters and have thought about the distances and sizes of the earth, the sun, the moon, and the universe in its entirety will accept them due to my argument, and that is why I believed that you might enjoy having brought it to your attention.

2 Introduction

"The Sand Reckoner" might be the best introduction to ancient science:

1. It is addressed to the King of Syracuse, so may be the first research–expository paper ever written.

2. In its goal of addressing "innumeracy", it is relevant to a modern audience.

3. It contains many details about ancient astronomy, but also motivates them by presenting them in the context of solving a specific problem.

4. The first known example of an astronomical experiment.

5. The first example of psychophysics, the study of human beings as measuring instruments.

6. Faces the problem of naming and manipulating large numbers without using modern notation.

The paper addresses the problem of "innummeracy" in ancient Greece, in particular, they did not believe that there were numbers great enough to describe the amount of sand. This belief was common and "sand" was synonymous with "uncountable." In order to rectify this situation, Archimedes wrote the paper to a non–mathematical audience, King Gelon, King of Syracuse, and the paper can therefore be described as the first research expository paper.

Archimedes sets for himself the task to name a number larger than the number of sand not just on a beach, or even on all of the earth, but the idealized question of naming a number that would be larger than the number of sand that could fill up the whole universe.

The reason for this generalization is clear. By taking the largest amount of sand possible, one can give an upper bound that will apply to any possible amount of sand, and thus solve the problem completely.

In order to solve the problem, Archimedes needs to make some physical assumptions, and then apply mathematical techniques to them. The paper thus has two themes: (a) physical assumptions based on observational data, (b) mathematical analysis.

Since Archimedes was a mathematician, the mathematical analysis is completely rigorous, but this is clearly not possible for the physical part of the paper. This part of the paper is also written in two different styles. The experiments that Archimedes is able to perform himself are analyzed with extreme precision, much more than the other data will allow, while experiments that he merely reports are overestimated by a factor of 10. This last strategy is successful in that he actually overestimates the distance to the sun, even though contemporary estimates of the distance to the sun were much smaller and estimating this distance is quite difficult.

In trying to estimate the amount of sand that could fill the universe, Archimedes must first address the definition of the universe in order to estimate its size. Archimedes states that, for the purposes of the paper, he will adopt the heliocentric theory of Aristarchus of Samos. The reason for this is that Archimedes, in order not to have his result superseded, needed to find the largest model of the universe. He chose the heliocentric theory because it requires the stars to be much further away in order to avoid stellar parallax. This has great historical interest because it is one of the only references to Aristarchus' heliocentric theory, as the work itself is lost.

Archimedes notes that Aristarchus was not precise about how far the stars are from the earth, so he makes the assumption that the distance to the stars is in the same to the radius of the earth's orbit as the radius of the earth's orbit around the sun is to the radius of the earth. The reasoning behind such an assumption is that since observers on earth do not notice the sun moving locally due to the earth's rotation (solar parallax) the ratio of the earth's radius to the distance to the sun is large enough to make such an effect unobservable. It follows that putting the stars' distance with respect to the earth's rotation in the same ratio should exclude any apparent motion of the stars. In effect, Archimedes is saying that stellar parallax equals solar parallax. Symbolically, this would be written as

$$\frac{r_u}{d_s} = \frac{d_s}{r_e},$$

where r_u is the radius of the universe, i.e., the distance to the stars, d_s is the distance to the sun, and r_e is the radius of the earth. In order to compute r_u, Archimedes must give values for r_e and d_s, so he proceeds to quote estimates of the earth's radius, and in order to estimate d_s, the relative sizes of the earth, moon, and sun.

The contemporary estimates for the circumference of the earth were quite accurate, for example, Eratosthenes' celebrated measurement of yielded approximately 25,000 miles which is within 1,000 miles of the actual figure. Since Archimedes has not performed this experiment, he overestimates this by a factor of ten and arrive at a radius of about 40,000 miles??.

The problem of estimating the distance to the sun is quite difficult and accurate estimates were not obtained until the 18th century. The method that Archimedes uses is to first estimate the size of the moon relative to the earth, then the size of the sun relative to the moon. Once an estimate of the size of the sun is given, then the distance to the sun can be estimated by measuring the angular size of the sun, as seen on earth, and using some equivalent form of a trigonometric formula.

Aristarchus of Samos gave a method for estimating the sizes of the moon and sun, but his estimate of the sun is much smaller than the actual value. This method first estimates the size of the moon, which can be done fairly well by estimating the size of the earth's shadow on the moon during a lunar eclipse. This shows that the moon is at least 1/3 the size of the earth. Next, Aristarchus looked at the angle that the moon and sun make when the moon is exactly half illuminated by the sun. In modern notation, if this angle is θ, then $d_m/d_s = \cos\theta$, where d_m is the distance to the moon. Now the actual value of θ is $89°50'$, which is indistinguishable from $90°$ using ancient techniques, but more importantly, deciding when the moon is half full is too difficult to make this measurement with anything close to this level of precision. However, one can conclude that $d_s > 20d_m$ but Archimedes overestimates this to be $d_s > 30d_m$. Since the sun and moon have the same angular diameter with respect to a terrestrial observer, as seen during solar eclipses, it follows that $r_s > 30r_m > 30/3r_e = 10r_e$, and the radius of the sun is at least 400,000 miles.

The next step is to compute the angular size of the sun. This is done with extreme care by Archimedes himself. Thus, he is very careful to note that in measuring the angular size of the sun one should take into account the size of the eye, as this will affect the answer slightly. This is of interest as it is the first example of the science of psychophysics, i.e., analyzing the human body as an instrument. Furthermore, he also takes into account solar parallax, in other words, the fact that his estimate of the distance to the sun is taken from a measurement on the surface of the earth, while the actual distance that he is interested in is taken from the centre of the earth. This is also the first known example of solar parallax being taken into account. As an example of the inconsistencies of the paper, note that this adjustment for solar parallax contradicts his previous implicit assumption that solar parallax is negligible.

From this one can say that apart from the overestimates for the size of the earth and sun, and the distance from the earth to the sun, Archimedes is actually computing the order of magnitude of the answer. In other words, except for two steps in the computation, the estimate will be correct within a factor of ten.

Once Archimedes has collected the physical data, he then develops a system for naming numbers, since the contemporary Greek system only went up to ten thousand (a myriad) and naming number larger than ten million (a myriad myriad) was cumbersome. His method is to essentially use base 10,000 which allows him to name powers of 10,000 up to 10000^{10000}. Archimedes then considers the sequence of powers of 10 and essentially states the formula $10^a 10^b = 10^{a+b}$.

The paper ends with a very longwinded description of how to use the estimates of the size of a grain of sand, the earth, and the distance from the earth to the sun in order to get the upper bound 10^{63} for the number of grains of sand in the universe. The main reason for the length is that Archimedes only allows himself to use the law of exponents in the form $10^a 10^6 = 10^{a+6}$.

Thus instead of computing how many diameters of a grain of sand will be an upper bound for the diameter of the cosmos then cubing this number to find the ratio of volumes, he considers separately each increase in diameter by a factor of one hundred $= 10^2$, and then multiplies the ratio of volumes by 100 myriad ($= 10^6$).

3 The problem

The Greek title of the paper is $\psi\alpha\mu\mu\iota\tau\eta\sigma$ meaning having to do with sand [26]. The Latin translation is *Arenarius* which also means having to do with sand, but can be understood as meaning arithmetic, as these were done on sand. From this perspective, the Latin title might be the most appropriate.

The paper is addressed to King Gelon, son of King Hieron, who were co–rulers of Syracuse, the city where Archimedes lived, so one way of estimating the date of the paper would be to know when Gelon acceded to the throne.

Archimedes refers to the belief among his contemporaries that sand was infinite or uncountable. This is confirmed in a passage from Pindar's *Olympic Ode II* [28]: "...sand escapes counting" ($\psi\alpha\mu\mu o\sigma$ $\alpha\rho\iota\theta\mu o\nu$ $\pi\epsilon\rho\iota\pi\epsilon\phi\epsilon\upsilon\gamma\epsilon\nu$). Moreover, the word $\psi\alpha\mu\mu\alpha\kappa o\sigma\iota o\iota$ (sand–hundred) was used to denote a very large number, as in the modern English word "zillion." The exaggerated form $\psi\alpha\mu\mu\alpha\kappa o\sigma\iota\gamma\alpha\rho\gamma\alpha\rho o\iota$ (sand dune hundred) appears in the opening lines of Aristophanes' play *The Archanians* [6, p.6] a typical English translation being [7, p. 101]:

> "How often have I chewed my heart with rage! My pleasures? Very few; in fact just four. My pains? Far more than all the grains of sand."

A similar statement appears in the *Iliad* IX, 385 [21]: "Not if his gifts outnumbered the sea sands or all the dust grains in the world would Agamémnon ever appease me..."

The uncountability of sand also appears 21 times in The Bible [33, p. 1179]. For example, in Genesis 32:12: "And thou saidst, I will surely do thee good, and make thy seed as the sand of the sea, which cannot be counted for multitude." It is a comment on the lack of impact of Archimedes' work that similar comments appear in the New Testament, e.g., Hebrews 11:12: "So many as the stars of the sky in multitude, and as the sand which is by the seashore innumerable."

Given the question of counting grains of sand, Archimedes immediately generalizes this question not just to the harder problem of counting the sand that can be seen on a single beach, or in Sicily, but to the entire surface and volume of the earth.

The language of Archimedes uses to express this is to consider "a mass made up of sand. . . as large as the mass of the earth, including in it all the seas and hollows of the earth filled up to the height of the largest mountain." In this way, Archimedes defines a sphere *and* its interior with radius the height of the highest mountain. This passage is slightly ambiguous since the term "filled–up" can be interpreted as filling–up with sand.

However, Archimedes does not stop with this harder problem but immediately goes on to find the *largest possible* amount of sand. His object seems to avoid having his estimate superseded by a larger example. This competitive approach is also seen in his sending incomplete or false proofs as a test to his readers and the (highly speculative) explanations that the difficulty of his Cattle Problem was in response to having his works superseded by Apollonius [35].

4 Physical assumptions

4.1 The size of the universe

In order to fill the universe with sand, Archimedes has to give a concrete definition of the universe. In the paper, he claims that astronomers defined the universe to be the sphere with center at the centre of the earth and radius the distance from the centre of the earth to the centre of the sun.

Now, since the stars are so far away that only their angular separation is observable to the naked eye, they all seem to be the same distance from an observer which gives the starry sky a spherical appearance. It was therefore natural for ancient astronomers to believe that the "fixed stars," i.e., all stars but the sun, moon, and planets, were on one sphere.

However, since the sun, moon, and planets, move relative to the stars, these were eventually thought of as being closer to the earth than the rest of the stars. For example, it is clear that, since the moon obscures the sun during a solar eclipse, it is is closer than the sun. Since the sun and moon retain approximately the same angular size, these were assumed to travel on their own spherical shells, as were the five known planets, Mercury, Venus, Mars, Jupiter, and Saturn. However, the rate of rotation of these bodies was assumed to be proportional to their distance to the earth, so that the moon was closest, and Saturn the furthest. In, fact in *Timaeus*, [29] Plato gives the order as: moon, sun, Venus, Mercury, Mars, Jupiter, Saturn, stars (there is a problem ordering Mercury, Venus, and the sun, since their periods of rotation are roughly equal).

The theory of spherical shells was refined by Eudoxus (ca. 408B.C.–355B.C.) to explain the non–circular motions of the sun, moon, and planets (for example, the fact that annular solar eclipses sometimes occur implies that the sun or the moon does not have a circular orbit).

Furthermore, in *Meterologica* Aristotle states [17, p. 331]:

> "Besides, if the facts as shown in the theorems of astronomy are correct, and the size of the sun is greater than that of the earth, while the distance of the stars from the earth is many times

> greater than the distance of the sun, just as the distance of the sun from the earth is many times greater than that of the moon, the cone marking the convergence of the sun's rays (after passing the earth) will have its vertex not far from earth, and the earth's shadow, which we call night, will therefore not reach the stars, but all the stars will necessarily be in the view of the sun, and non of them will be blocked out by the earth."

For the above reasons (note that Archimedes refers to Eudoxus in this paper) it is not at all clear why Archimedes would claim that the sun's orbit was the limit of the universe. Moreover, even if astronomers used this definition in a purely semantic way, it would not serve Archimedes' purpose in the paper, as it would still be conceivable to fill up with sand past the sun all the way to the fixed stars. In fact, Archimedes avoids this problem by using a different model of the universe.

4.2 The earth is round

The concept of "centre of the earth" mentioned by Archimedes clearly assumes the fact that the earth is not flat. Indeed, the fact that the earth is round had been known for centuries before Archimedes, namely by Pythagoras (ca. 572B.C.–500B.C.) who is believed to be the first person to have proved this. Arguments were later written by Aristotle [5, XIV, p 252], who noted that the earth makes a round shadow during an eclipse of the moon. The shadow is known to be the earth's because a lunar eclipse only occurs on a full moon, when the sun and moon are observed to be in in opposition, i.e., in a straight line and at opposite ends of the sky. This argument was criticized by Neugebauer [27, p. 1093], who pointed out that a nonspherical object can also make a circular shadow. Another objection is the phenomenon, observed by ancient astronomers, of "paradoxical" lunar eclipses occurring at dusk in which both the sun and the fully eclipsed moon are simultaneously visible. This would contradict that the moon, earth, and sun are in alignment, and that it is the earth's shadow that causes the eclipse. One ancient astronomer, Cleomedes (ca. 150B.C.) actually gave the correct explanation, namely that it was caused by the refraction of the earth's atmosphere which bent the light and caused the setting sun to be seen when it was actually below the horizon so ancient astronomers also knew that the sunset is an optical illusion), see the translation by Heath in [18, p. 162–166] of Cleomedes argument in *De Motu Circulari Corporum Caelestium* [11]. The amount of distortion of a celestial object at the horizon by refraction is now known to be about $34'$ or about the same size as the angular diameter of the sun or moon [16, p. 95].

Better reasons for the spherical shape of the earth are given by Ptolemy [30, I.4]:

> "*That also the earth, taken as a whole, is sensibly spherical,*
>
> Now, that also the earth taken as a whole is sensibly spherical, we could most likely think out in this way. For again it is possible to see that the sun and moon and the other stars do not rise and set at the same time for every observer on the earth, but always earlier for those living towards the orient and later for those living towards the occident. For we find that the phenomena of eclipses taking place at the same time, especially those of the moon, are not recorded at the same hours for everyone–that is, relatively to equal intervals of time from noon; but always find later hours recorded for observers towards the orient than for those towards the occident. And since the differences in the hours is found to be proportional to the distances

> between the places, one would reasonably suppose the surface of the earth spherical, with the result that the general uniformity of curvature would assure every part's covering those following it proportionately. But this would not happen if the figure were any other, as can be seen from the following considerations.
>
> For if it were concave, the rising stars would appear first to people towards the occident; and if it were flat, the stars would rise and set for all people together and at the same time; and if it were a pyramid, a cube, or any other polygonal figure, they would again appear at the same time for all observers on the same straight line [face]. But none of these things appears to happen. It is further clear that it could not be cylindrical with the curved surface turned to the risings and settings and the plane bases to the poles of the universe, which some think more plausible. For then never would any of the stars be always visible to any of the inhabitants of the curved surface, but either all the stars would both rise and set for observers or the same stars for an equal distance from either of the poles would always be invisible to all observers. Yet the more we advance towards the north pole, the more the southern stars are hidden and the northern stars appear. So it is clear here the curvature of the earth covering parts uniformly in oblique directions proves its spherical form on every side. Again, whenever we sail towards mountains or any high places from whatever angle in whatever direction, we see their bulk little by little increasing as if they were arising from the sea, whereas before they seemed submerged because of the curvature of the water's surface."

Note that Ptolemy's is very careful to show that the earth's curvature is the same in all directions.

4.3 Aristarchus and the heliocentric theory

Most people believe that it was Copernicus who first proposed the heliocentric theory of the universe, so it might come as a surprise that this theory was proposed some 1800 years earlier by Aristarchus of Samos. Though some have claimed that Copernicus was aware that Aristarchus had a first claim on this theory, this view has recently been challenged by Gingerich in his paper *Did Copernicus owe a debt to Aristarchus?* [15]. A semi heliocentric theory, i.e., one where Mercury and Venus orbit the sun which orbits the earth, is claimed to have been proposed by Heraclides Pontus (ca. 388B.C.–315 B.C.). This view has been challenged by Neugebauer [27, p. 320].

Aristarchus' work on the heliocentric theory has been lost and is only known through references to it such as this one. Another works of his have survived [4] and the ability that he shows in this paper implies that his heliocentric theory was based on sound theoretical principles [17], and it should be noted that Archimedes takes it seriously in this paper.

The reason why Archimedes mentions Aristarchus' theory is explained by the end of the last sentence:

> "...and that the sphere of fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface."

The point is that in a heliocentric theory, the stars must be much farther away than in a geocentric theory (see [23] for an analysis of the philosophical implications of this "larger" universe). Thus Archimedes went shopping around for different theories of the universe trying to find the one with largest size so that his computation would not be superseded

The reason that the heliocentric theory leads to a larger universe is the phenomenon of *parallax*, i.e., that an object which is quite close seems to be at a different angle if viewed from a slightly different position. One example is if you hold an object a foot away from your eyes, then it seems different when viewed from one eye or the other, another one is that when you are moving, object that are close seem to move while faraway object don't (e.g., the moon "follows" you when you walk). So in a heliocentric model, if the stars were too close, i.e., relative to the distance of the earth to the sun, then in a six month period, the angle at which a star would be seen would change a lot. To avoid the parallax problem, the stars have to be so far away relative to the distance between the earth and the sun that is no longer observable.

Aristarchus' resolution of the parallax problem is to make the ratio of distance of the stars to radius of the earth's orbit around the sun very much larger, and essentially infinitely large since the ratio he gives, as referred to by Archimedes, is the ratio of a surface to a point.

Clearly, this will not do, first of all, on logical grounds that the ratio of a surface to a point makes no sense, and secondly, an infinite universe *would* contain an infinite amount of sand. Archimedes therefore has to give a meaningful interpretation of Aristarchus' theory of the size of the universe, and he continues:

> "Now it is easy to see that this is impossible; for, since the centre of the sphere has no magnitude, we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must take Aristarchus to mean this: since we conceive the earth to be, as it were, the centre of the universe, the ratio which the earth bears to what we describe as the 'universe' is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of fixed stars."

The justification for Archimedes' amendment to Aristarchus' theory is apparent from the phrase: "...since we conceive the earth to be, as it were, the centre of the universe..." Recall that Archimedes is using the word "universe" to mean the sphere with earth at the centre and radius the distance from the earth to the sun, so he is saying that from the vantage point of earth, it looks like the sun is going around in a perfect circle around the centre of earth. It is reasonable to infer that one should not expect to see a variation in the sun's position depending on where the sun is viewed from different places on the earth (solar parallax). Archimedes assumption is thus

$$\text{Stellar Parallax} = \text{Solar Parallax}.$$

It then follows that if the ratio of the radius of the universe to the ratio of the earth's orbit around the sun is the same as the ratio of the earth's orbit around the sun to the radius of the earth, then there will not be any stellar parallax problem either (though Archimedes contradicts this later on when he takes into account solar parallax, see below).

Estimating solar parallax is actually quite easy, as is seen in Figure ??. This shows that the parallax of a distant object as seen from one end of the earth versus the other end of the earth is exactly equal to the angular size of the earth seen from the distant object. This has the immediate consequence that:

Maximum solar parallax is equal to the angular size of the sun divided by the ratio of the sun's diameter to the earth's diameter.

Archimedes will later assume that: (a) The angular size of the sun is about 1/2 of a degree. (b) The sun is about ten times larger than the earth. This implies that solar parallax is about $3'$, or 1/20 of a degree (and the same value for stellar parallax over a six month period) which would probably not be observable without a telescope. (The actual value is about $17''.6$ on average [16, p. 105] and the smaller figure corresponds to Archimedes' underestimation of the size of the sun.)

One could ask whether it would have been valid to assume that the size of the universe should be taken to be in the ratio of the distance of the moon relative to the size of the earth. Since Archimedes probably assumed that the angular size of the moon was 1/2 of a degree and that the moon was about 1/3 the size of the earth, it would follow that the maximum parallax would be 3/2 of a degree which might be observable (the actual value is about 2 degrees, on average). This would therefore not make a good choice.

4.4 The perimeter of the earth

It is now well known that Eratosthenes (ca. 276B.C.–??) made a very good estimate of the earth circumference [8] [24], and [34, p. 267] for a more contemporary report by Cleomedes ca. 150 B.C.

Eratosthenes' procedure was as follows: He noted that on the summer solstice the sun made no shadow in the ancient city of Syene (the modern Asswan, Egypt). However, at the same time, in Alexandria, the shadow at noon made an angle of about 7 degrees, or 1/50 of a full circle. Since he estimated the distance from Syene to Alexandria to be $5,000$ stades, this gave a circumference of $250,000$ stades (note that Ptolemy's argument that the earth is spherical is required since Eratosthenes' measurement gives only a circumference in the North–South direction).

In order to check the accuracy of Eratosthenes' measurement, one must convert stades into modern units. This poses a problem, since there is no agreement as to the length of this measure. It is given as about 600 feet or about 200 meters in [20], where a *stadium* is defined to be the length of an Olympic stadium (or track). Boyer [9] uses 1 stade$\approx$ 1/10 mile, while Eves [13] gives 1 stade$\approx$ 559 feet. Heath [19] cites Pliny as giving 1 stade as 516.73 feet.

Heath's definition of the stade results in a circumference of $24,662$ miles, which is within 50 miles of the actual figure. This was actually computed using the estimate of $252,000$ stade adopted by Hipparchus and Theon of Smyrna, which is more convenient since this number is divisible by 60.

This number seems a little too accurate, and other writers have given the estimate to be $29,000$ miles using 1 stade being between 1/7.5 and 1/10 of a Roman mile [10, p. 154].

It is important to note that Eratosthenes' measurement is actually independent of units of measurement since it gives the circumference of the earth as 50 times the distance from Syene to Alexandria. Thus any subsequent estimate can be made by estimating this distance directly. For example, this could have been done by the contemporaries of Christopher Columbus when estimating the distance from Spain to China.

In any case, it is clear that $300,000$ stades is an overestimate for all known definitions of a stade, and anyway, Archimedes covers himself by overestimating by a factor of ten.

Remark. It should be emphasized that Eratosthenes' computation is simplified by the fact that Syene lies exactly on the tropic of Cancer. This means that on June 21 at high noon, the sun is directly overhead. If another city not at this latitude were chosen, then a comparison of *two* angles of shadows would have been required. More importantly, this leads one to suspect that it *because* Syene is on the tropic of Cancer that

Eratosthenes though up this method of measuring the circumference. The *qualitative* difference in shadows led him to consider a measurement of the *quantitative* difference in shadows.

4.5 The sizes of the earth, moon, and sun

Archimedes assumes that the earth is bigger than the moon and that the sun is bigger than the earth. Both of these can be explained fairly easily, thought the second statement requires a new idea and a physical experiment.

To see that the moon is smaller than the sun, recall that in a solar eclipse the moon barely covers the sun (and sometimes its angular diameter can be smaller, as in an annular eclipse) and that it is closer than the sun, since it obscures it during a solar eclipse.

Next, it can be seen that the earth is greater than the moon. For if the earth were the same size or smaller than the moon, then a lunar eclipse, as seen from the moon, would appear to be exactly the same as a solar eclipse does from the surface of the earth, and so the shadow of the earth would be quite small. But in fact, the earth shadow during a lunar eclipse covers the whole moon.

To show that the sun is greater than the earth requires new ideas that show the following:

(a) The moon is at least $1/4$ the size of the earth.

(b) The sun is at least 8 times the size of the moon.

From this it follows that the sun is at least $8/4 = 2$ times the size of the earth.

These ideas were introduced by Aristarchus in his paper *On the sizes and distances of the sun and moon* [4] who used the following observations

(i) To estimate the size of the earth's shadow on the moon

(Aristarchus estimated it at twice the moon's size, while the true figure is slightly less than three).

(ii) To observe the angle between the moon and the sun when the moon is exactly half full, i.e., when the line between the moon and the sun makes a $90°$ angle with the line between the observer and the moon. (Note that this second method requires the knowledge that the moon's illumination comes from the reflected light of the sun, a fact that had been discovered by Anaxogoras (ca. 500B.C.–428B.C.) two centuries earlier.)

The first observation gives a lower limit on the size of the moon, while the second gives an estimate of the ratio of the moon's distance to the sun's distance, and so of their relative sizes.

As will be seen in the next section, the size of the earth shadow on the moon implies that the moon has a radius at least $1/3$ of the earth's. Moreover the actual angle that the moon and sun make when the moon is half full is $89°50'$, so it should be possible to estimate this as being bigger than $83°$, which has cosine less than $1/8$. In other words, such an estimate shows that the sun is at least 8 times the size of the moon.

Aristarchus used his method to try to get much more precise information about the sizes of the sun and moon. Unfortunately, there are many problems with this method. As noted above the actual angle is more like $89°50'$, which is indistinguishable from $90°$, for example, the horizontal refraction discussed above is $34'$, so that this effect alone is of the same order as what needs to be observed, at shallow angles at least.

The other problem is that the exact time when the moon is half full is extremely hard to determine.

Aristarchus, gave the grossly inaccurate figure of 87° which led him to get severely underestimate the size and distance of the sun. It is now believed that Aristarchus' paper is meant more as an application of his correct idea and as a mathematical exposition of how it could work [17].

Strangely enough, Aristarchus does not give an explicit estimate of the distance to the sun, but his method does give a *lower* bound, which is essentially the one given by Archimedes. He then states

> "It is true that, of the earlier astronomers, Eudoxus declared it to be about nine times as great, and Pheidias, my father twelve times, while Aristarchus tried to prove that the diameter of the sun is greater than 18 times but less than 20 times the diameter of the moon."

This passage is interesting in regards to the biographical information that it reveals about Archimedes' father. The computation of Aristarchus that is alluded to is his paper *On the sizes and distances of the sun and moon*, as mentioned above.

The general principle by which the size of the moon can be estimated is the following: Under the assumption that the sun is much farther away than the moon, the shadow of the earth is roughly the same size as the earth, at least when it obscures the moon. Thus the ratio of the moon's diameter to the shadow of the earth during a lunar eclipse should roughly be the ratio of the moon's diameter to the earth's diameter.

The method of Aristarchus is simply a precise way to compute this. What he shows is:

Theorem of Aristarchus. *Let s be the radius of the sun, ℓ the radius of the moon, and t the radius of the earth. Furthermore, let u be the ratio of the radius of the shadow of the earth to the radius of the moon. Finally, let ψ be the angle between the moon and the sun, as seen from the earth when the moon is exactly half illuminated. Then,*

$$\frac{\ell}{t} = \frac{1+\cos\psi}{1+u}, \qquad \frac{s}{t} = \frac{1+\cos\psi}{\cos\psi(1+u)}.$$

The first formula has the interesting feature that there is not too much dependence of ψ, it contributes at most a factor of two. In fact, ψ will be very close to 90°, so that this formula will be

$$\frac{\ell}{t} \approx \frac{1}{1+u},$$

so only the value of u matters. Aristarchus believed that $u = 2$ so that $\ell \approx .33t$. Ptolemy later gave the value $\ell/t = 2\,\frac{3}{5}$.

The second formula is similar except for the $1/\cos\psi$ term. Since ψ is close to 90° (Aristarchus took it to be 87° and the actual value is about $89°50'$), $\cos\psi$ is small, so that there is a lot of instability in the value of s/t. This makes it clear that no precise calculation of the sun's size is possible using such methods.

These methods can also give estimates for the distances of the sun and moon because the common ratio $\ell/L = s/S$ is $\sin\theta$, where θ, the angular diameter of the sun, can be measured physically. Aristarchus first reported $\theta = 2°$, but this was quickly corrected to $1/2°$, as is the subject of the next section.

Proof of Aristarchus' theorem: One starts with the fact that the moon has almost the same angular size as the sun, as viewed from earth,

$$\frac{\ell}{L} = \frac{s}{S} = \alpha = \sin\theta,$$

where θ is the angular size of the moon or sun.

Now consider a full lunar eclipse. Let D equal the distance of the center of the earth to where the shadow of the earth meets at a point. By similar triangles

$$\frac{D}{S} = \frac{t}{s-t},$$

and substituting $s = s/\alpha$ gives

$$D = \frac{ts}{\alpha(s-t)}.$$

By similar triangles, one also gets

$$\frac{D}{t} = \frac{D-L}{u\ell}$$

so that

$$D = \frac{\ell t}{\alpha(t-u\ell)}.$$

Equating these two values for D gives

$$\frac{ts}{\alpha(s-t)} = \frac{\ell t}{\alpha(t-u\ell)}.$$

Now let $L/S = \beta = \cos\psi$, then also $\ell/s = \beta$ (so $s/\ell = 1/\beta$) by the lunar eclipse observation. Substituting this into the last equation and dividing out the common factor of t/α gives

$$\frac{\ell/\beta}{\ell/\beta - t} = \frac{\ell}{t-u\ell}.$$

This can be simplified to

$$\frac{1}{\ell-\beta t} = \frac{1}{t-u\ell},$$

which is the same as

$$\frac{1}{\ell/t-\beta} = \frac{1}{1-u\ell/t}.$$

This can be solved for $x = \ell/t$:

$$\frac{1}{x-\beta} = \frac{1}{1-ux} \Longrightarrow 1-ux = x-\beta \Longrightarrow x = \frac{1+\beta}{1+u}.$$

5 Experiments

I have tried to reproduce Archimedes' experiment to measure the angular diameter of the sun. My first attempt was on March 19, 1997, when I went to Venice Beach, CA, with a meter long ruler, and some cylindrical weights from a set of chemistry weight. Between 6p.m. and 6:05p.m., I put a cylinder of diameter 9mm. and height 9mm on the ruler, pointed in the direction of the setting sun. The cylinder seemed to be about 820mm, but there seemed to be some portion of the sun visible from about 880mm. This was on a second attempt. On the first attempt, I got distances of 810mm and 950mm, respectively.

Some preliminary remarks from this are:

1 Archimedes either had help, or else his cylinders were very small (at most 5mm in diameter) since anything larger than 5mm would be 50cm away, and so too far to move by yourself while looking at the sun.

2 The weights did not cast a clear shadow, so determining this angle using shadows did not seem practicable.

Figure 5.0.1[p]: Lower bound on the size of the sun

3 The stand Archimedes refers to had to let the ruler rotate, since in the second estimate the sun must be seen on both sides of the cylinder, so the direction of the sun has to be precise.

4 The day has to be very clear, as any clouds seem to distort the sun when it is on the horizon.

5 Even when the sun was on the horizon, its brightness would still cause light to seem to appear from the sides of the cylinder.

One can conjecture that Archimedes could have done this experiment alone as his exposition in the paper leads one to believe that he would not trust anyone else to do it for him. Perhaps one way would have been for him to use a system of pulleys, since such methods are attributed to him elsewhere, for example by Plutarch.

Archimedes then does further experiments in order to compensate for the fact that the angle the sun makes with the eye does not have its vertex at the eye, since the eye actually sees from an area, not from a point. In order to compensate for this, he tries to compute the diameter of the pupil by taking two cylinders, one white and one normal, and putting one as close to the eye as possible so it occludes the white one.

This experiment has a number of problems associated with it.

(i) The experiment requires a cylinder that is of about the same size as the pupil, which requires knowing its size in the first place.

(ii) The size of the pupil varies according to light conditions.

In fact, the bounds Archimedes wants can easily be achieved without any reference to the pupil. A lower bound on angular size of the sun can be done as follows:

Take a cylinder and place it so that you can just see the sun on its edges, then take a smaller cylinder and place it so that it just covers the other cylinder. The angle between tangents to the cylinders will be a lower bound. Similarly one gets an upper bound:

Take a cylinder and place it so that it just covers the sun, then place a smaller cylinder so that one can just see the edges of the larger cylinder. The angle between tangents to the cylinders will be an upper bound.

Note that this method uses fewer cylinders. In any case, Archimedes' digression makes him the first person known to take account human physiology in a physical measurement, the study of which later became known as psychophysics, a field developed by Hermann Helmoltz (1821–1894).

This fact does not seem to be known among experimental psychologists and there are only a few papers written on this subject [25] [32].

Other criticisms of this experiment are that

Figure 5.0.2[p]: Lower bound on the size of the sun

Figure 5.0.3[p]: Refraction magnifying size

(a) The experiment can also be done with the moon, since solar eclipses show that the sun and moon have the same angular size. The advantage of this is that the experiment can be done on a less bright object (for example, on a partial moon).

(b) Archimedes does not address the question of the moon illusion, in other words, that the sun and moon appear to be larger when on the horizon. This is important because showing that this is in fact an illusion *requires* an accurate measurement. In fact, it is easy to show that the sun is not actually closer, for example, since it sets at different positions on its orbit at different times of the year, but this does not explain whether the illusion is caused by something physical, e.g., an atmospheric distortion, or is purely a product of human perception.

In fact, there is a reference to Archimedes explaining the moon illusion as a consequence of atmospheric refraction in a commentary of Theon of Alexandria on Ptolemy's *Almagest* which I have translated from [3, Vol. 4, p. 207] (also [2]):

> To refute the opinion that celestial bodies appear larger when they are near the horizon because they are seen from a smaller distance, Ptolemy proposes here to analyze a phenomenon of this kind and to show that it does not occur because of the distance between earth and sky but that due to the very humid emanations that surround the earth, the visual field encounters a body of air that is denser and that the rays going to the eye through the air are refracted and thus make the apparent angle at the eye larger as was shown by Archimedes in his treatise *On catoptrica,* where he says that objects submerged in water also seem larger, and the more so the deeper... Let the lines $E\Theta A$ and EKB, oriented by refraction, as in Archimedes in his treatise *On catoptrica,* to the points A and B as we have said.

If this is true, then it has a serious impact on Archimedes' experiment since its inclusion in the *Sand Reckoner* may have been related to work on explaining the moon illusion. This would explain why he would go to so much pains to get an accurate measurement, even compensating for the size of the eye, as he might be comparing this measurement with a future one of the sun at its zenith.

Note that Ptolemy later dismissed this explanation and claimed that the moon illusion was due to human perception [31, p. ??].

6 Solar parallax

The next part of the paper is devoted to a consideration of the effect of solar parallax on the estimation of the distance of the sun. Since Archimedes has previously stated that this quantity is negligible, one can only wonder why he takes it in to account later. One possibility is that he wants to make his method conceptually correct. Another reason might be related to the possibility stated above, that Archimedes was trying to explain the moon illusion. In this case, he would require an accurate measurement (this is consistent with his computation, see below). One can also conjecture that the inclusion of a parallax computation while ignoring the moon illusion in the paper is a subtle joke-namely, that the sun looks largest when it is actually farthest from the observer.

One sees that in Archimedes estimation of solar parallax, his final estimate is $s+r$ for the quantity $\sqrt{s^2+r^2}$, i.e., the error introduced by solar parallax is bounded by r. Now the upper bound $s+r^2/(2s)$ shows that this error is actually bounded by $r^2/(2s)$ which is smaller by a factor of $r/(2s)$. Since Archimedes assumes that s is at most $10000r$, even taking s as more than $1000r$ shows that Archimedes is overestimating the parallax error by a factor of 1000. This very poor estimate for a negligible correction can be explained assuming that Archimedes intended to make another observation of the sun at its zenith because in that case the value $s+r$ gives the exact stellar parallax correction and Archimedes might have wanted to introduce the same correction term for both observations. This gives further evidence that Archimedes was concerned with the moon illusion.

7 Large numbers

7.1 The current system

Recall that a number in base k is written as $a_0+a_1k+a_2k^2+\cdots+a_nk^n$, where $0 \leq a_0,\ldots,a_1 < k$. Modern notation uses base 10 for writing numbers, but the way English names of numbers are actually spoken is base 1000 in the following sense: A large number such as $2^{40} = 1,099,511,627,776$ is said to be "one trillion ninety nine billion five hundred and eleven million six hundred and twenty seven thousand seven hundred and seventy six." The sequence "million, billion, trillion,..." represents the number 1000^{n+1} as "latin(n)–illion" where latin(n) is the Latin name for n.

The origin of this system dates back to Italy where *millione* (= great thousand) was used to denote a thousand thousand. Around 1484 N. Chuquet used billion, trillion,...,nonillion which appeared in print in a 1520 book of Emile de la Roche. These numbers denoted powers of a million, i.e., a billion was a million million, a trillion was a million billion, etc. However, around the middle of the 17th century billion = thousand million, trillion = thousand billion,..., started to be used in France. This is the system now used in the United States, but the original system is still used in Great Britain and Germany.

Dictionaries only list numbers up to a vigintillion, and since this corresponds to Latin for twenty, in American notation it is $1000^{21} = 10^{63}$.

7.2 Naming large numbers

Here is, essentially, Archimedes' method for naming large numbers. Start with a named number N, then consider powers $N^2, N^3, \ldots$ and introduce an auxilliary terms such as *order* or *period*. One then calls N the unit of the first order, N^2 the unit of the second order, and so on. This allows one to name numbers, e.g., if $N = 10000$, then $1093N^2 + 3511N + 1$ would be "one thousand and ninety three units of the second order and three thousand five hundred and eleven units of the first order and 1."

One runs out of name of orders at the Nth order, which is N^N. It follows that introduction of a new symbol such as order or period allows one to name numbers up to N^N. Note, however, that naming orders uses *ordinals,* i.e., first, second, third,..., so going up to N^N requires one to have names for the ordinals up to N as well.

Upon reflection it is seen that the English language system uses this system, where the new symbol is the suffix "illion," i.e., latin(n)–illion means a number of the nth order. However, it does not seem that the latin prefix corresponds to an ordinal.

In [22] D.E. Knuth gives a more efficient nomenclature for large numbers along the following lines. Given names for all numbers up to N, one can name all numbers up to N^2 by using $aN + b$, where $a, b \leq N$. For example, one says "nineteen hundred and ninety seven" for 1997 ($N = 100$). It follows that a name for a hundred hundred makes sense, so that the next name in the system is the myriad. This allows one to count up to a myriad myriad, and the new name for this is a *myllion.* Similarly, a *byllion* is a myllion myllion and in general latin(n)–yllion is latin($n-1$)–yllion^2, i.e., $10^{2^{n+2}}$.

This system is more efficient at naming large numbers than one where a new name is given for each power of some number, however it has the disadvantage that one cannot easily recognize the exponent of the base from the name of the number. The point is that large numbers are rarely named exactly and are most often used for rough estimates, so that the power of ten is the most important information contained in the number.

7.3 The Greek system

In Archimedes' time, the system used by Greeks was hardly adequate to express any number larger that a hundred million. The Greeks had an alphabetic method of writing numbers according to the scheme

$$\alpha = 1,\ \beta = 2,\ \gamma = 3,\ \delta = 4,\ \epsilon = 5,\ a = 6,\ \zeta = 7,\ \eta = 8,\ \theta = 9,$$

$$\iota = 10,\ \kappa = 20,\ \lambda = 30,\ \mu = 40,\ \nu = 50,\ \xi = 60, o = 70, \pi = 80,\ b = 90,$$

$$\rho = 100,\ \sigma = 200,\ \tau = 300,\ \upsilon = 400,\ \phi = 500,\ \chi = 600,\ \psi = 700,\ \omega = 800,\ c = 900.$$

The Greeks had to use three special symbols in in order to get all numbers less than a thousand. These are represented here as a, b, c, due to lack of fonts. The first letter is *digamma* or *stigma* in minuscule, the second is *koppa*, and the third is *san* or *ssade*, and later, in minuscule, called *sampi.*

The thousands were represented as a lower left stroke followed by one of $\alpha, \ldots, \theta$, or else by one of the capital letters $A, \ldots, \Theta$. Thus, 1234 would be written as $\overline{A\sigma\lambda\delta}$ or $\overline{{}_{,}\alpha\sigma\lambda\delta}$, where the overline is meant to indicate that this is a number, not a word.

Numbers over ten thousand would be expressed by writing the number of ten thousands over a μ, as ten thousand was called a myriad ($\mu\upsilon\rho\iota\alpha\delta\epsilon\sigma$).

The Greek system did not express numbers larger than a myriad myriad, "ten thousand times ten thousand," e.g., Daniel 7:10 and Revelation 5:11.

7.4 Archimedes' system

Archimedes begins the description of this system by noting that the Greek language already has names for numbers up to a myriad. He then observes that this allows one to name numbers up to a myriad myriad ($= 10^8$), as noted above. This is the largest number named in the contemporary Greek system and Archimedes uses it as the base of his system. In order to simplify the exposition, define $\Omega = 10^8$.

Archimedes calls the numbers up to M *first numbers* which Heath [1] calls *numbers of the first order.* Note that Archimedes' system expresses *ranges* of numbers, as opposed to naming a single number.

According to the above, introduction of the new symbol "numbers" which can be taken to mean "order of numbers" should allow Archimedes to name numbers up to Ω^Ω, and this is what he proceeds to do. The last number of the first order is Ω which Archimedes calls the *unit of second numbers,* and the numbers in the range Ω to Ω^2 are called the *second order.* In general the nth order will consist of the range Ω^{n-1} to Ω^n and Archimedes continues until the Ωth order which is the range $\Omega^{\Omega-1}$ to Ω^Ω. This last number is a myriad myriad to the myriad myriad power, as expected. As noted above, this requires being able to name all ordinals up to the Ωth, and this is possible, since these numbers are part of the Greek language.

Such large numbers are sufficient for any physical application, but in order to prove the power of his system Archimedes continues by introducing a second symbol, the *period.* Thus, the range of numbers defined up to the end of the Ωth numbers is called the first period, and the last number of the first period, which I call Π, is defined to be the unit of the second period, i.e., $\Pi = \Omega^\Omega$. According to the above, this could allow Archimedes to name numbers up to Π^Π, assuming that he had names for the ordinals up to Π. However, it will be seen that this is not the case.

In fact, Archimedes goes on to name the first order of the second period to be the range Π to $\Pi\Omega$, and the second order of the second period to be $\Pi\Omega$ to $\Pi\Omega^2$, and so on. In general, the nth order of the second period will be $\Pi\Omega^{n-1}$ to $\Pi\Omega^n$. One continues until the Ωth order of the second period which is the range $\Pi\Omega^{\Omega-1}$ to $\Pi\Omega^\Omega = \Pi^2$. This number is then taken to be the unit of numbers of the third period, and in the same way, the numbers Π^2 to Π^3 will be called the numbers of the 3rd period. In general, the numbers of the nth period will be the range Π^{n-1} to Π^n.

The Archimedean system can be described by

$$\text{First period}\begin{cases}\overbrace{1,\dots,\Omega}^{\text{1st numbers}},\\[1em] \overbrace{\Omega,\dots,\Omega^2}^{\text{2nd numbers}},\\[1em] \overbrace{\Omega^2,\dots,\Omega^3}^{\text{3rd numbers}},\\[1em] \quad\vdots\qquad\vdots\\[1em] \overbrace{\Omega^{n-1},\dots,\Omega^n}^{n\text{th numbers}},\\[1em] \quad\vdots\qquad\vdots\\[1em] \overbrace{\Omega^{\Omega-1},\dots,\Omega^\Omega=\Pi}^{\Omega\text{th numbers}},\end{cases},\quad \text{second period}\begin{cases}\overbrace{\Pi,\dots,\Pi\Omega}^{\text{1st numbers}},\\[1em] \overbrace{\Pi\Omega,\dots,\Pi\Omega^2}^{\text{2nd numbers}},\\[1em] \overbrace{\Pi\Omega^2,\dots,\Pi\Omega^3}^{\text{3rd numbers}},\\[1em] \quad\vdots\qquad\vdots\\[1em] \overbrace{\Pi\Omega^{n-1},\dots,\Pi\Omega^n}^{n\text{th numbers}},\\[1em] \quad\vdots\qquad\vdots\\[1em] \overbrace{\Pi\Omega^{\Omega-1},\dots,\Pi\Omega^\Omega=\Pi^2}^{\Omega\text{th numbers}}\end{cases},\dots,$$

and so forth until

$$\Omega\text{th period}\begin{cases}\overbrace{\Pi^{\Omega-1},\dots,\Pi^{\Omega-1}\Omega}^{\text{1st numbers}},\\[1em] \overbrace{\Pi^{\Omega-1}\Omega,\dots,\Pi^{\Omega-1}\Omega^2}^{\text{2nd numbers}},\\[1em] \overbrace{\Pi^{\Omega-1}\Omega^2,\dots,\Pi^{\Omega-1}\Omega^3}^{\text{3rd numbers}},\\[1em] \quad\vdots\qquad\vdots\\[1em] \overbrace{\Pi^{\Omega-1}\Omega^{n-1},\dots,\Pi^{\Omega-1}\Omega^n}^{n\text{th numbers}},\\[1em] \quad\vdots\qquad\vdots\\[1em] \overbrace{\Pi^{\Omega-1}\Omega^{\Omega-1},\dots,\Pi^{\Omega-1}\Omega^\Omega=\Pi^\Omega}^{\Omega\text{th numbers}}.\end{cases}.$$

Archimedes continues till the Ωth period which gives numbers in the range $\Pi^{\Omega-1}$ to Π^{Ω}. This last number is $\Pi^{10^8}=10^{8\cdot 10^{16}}$ is the largest one named by Archimedes and was written by him as: $\alpha\iota\ \mu\upsilon\rho\iota\alpha\kappa\iota\sigma\mu\upsilon\rho\iota o\sigma\tau\alpha\sigma\ \pi\epsilon\rho\iota o\delta o\upsilon\ \mu\upsilon\rho\iota\alpha\kappa\iota\sigma\mu\upsilon\rho\iota\sigma\tau\omega\nu\ \alpha\rho\iota\theta\mu\omega\nu\ \mu\upsilon\rho\iota\alpha\iota\ \mu\upsilon\rho\iota\alpha\delta\epsilon\sigma$.

This is translated by Heath as: "a myriad myriad units of myriad myriad numbers of the myriad myriad period." However, a more accurate translation would be: "A myriad myriad units of myriad–times myriadth numbers of the myriad–times myriadth period." Here, the term "myriad–times" represents the adverb for "myriad" as in "once, twice,. . . , myriad–times."

One sees that Archimedes stops at this number because he has failed to include names for the ordinals corresponding to his new numbers. The most evident reason is that Archimedes uses ranges of numbers which makes the naming of the corresponding ordinals difficult: What is the ordinal corresponding to the "unit of the second order"?

Moreover, modern English has accepted the conversion of any term denoting number into an ordinal by appending "th", as in $n \Longrightarrow n$th, but this is not necessarily the case with the ancient Greek language, where such a system must be more rigorous. For example, in English, Archimedes' largest number is: the unit of the second orderth period. Similarly, by "abuse of notation" one calls the nth orderth period the range $\Pi^{\Omega^{n-1}}$ to Π^{Ω^n}. It would then follow that the Ωth orderth period of numbers would be the range $\Pi^{\Omega^{\Omega-1}}$ to $\Pi^{\Omega^\Omega} = \Pi^\Pi$, as claimed. This last number could also be called the second periodth period of numbers. It would be

$$(10^{8\cdot 10^8})^{10^{8\cdot 10^8}} = 10^{8\cdot 10^{8(10^8+1)}}.$$

For his applications, Archimedes wants to apply the law of exponents to orders of numbers, so only needs a system to denote large power of 10 (D.H. Fowler [14, p. 225] remarks there is no evidence that Archimedes intended this system for anything other than this). As evidence, note that the term "myriad myriad myriad" appears, though it should have been called a "myriad second numbers" according to this scheme.

It is seen that his system is suboptimal because in naming *ranges* of numbers he effectively starts by naming the the unit as being the first in the list of powers, so that the unit of nth numbers is actually a M^{n+1}, instead of M^n. It follows that, according to this scheme, a unit of mth numbers times a unit of nth numbers will be the unit of $m + n - 1$ numbers.

Here one sees how Archimedes' definition can be improved. Instead of looking at ranges of numbers, consider how one might say a large number in terms of myriads, and define the order of a number as the largest number of times you repeat the word "myriad" consecutively. Thus, a myriad is the unit of the first order, as are number such as one hundred and twenty three myriads. Similarly, a myriad–myriad is the unit of the second order, a myriad-myriad-myriad is the unit of the third order, and so on. It is quite clear that such a nomenclature satisfies the law of exponents, as multiplication of units of orders is simply concatenation of the repeated "myriads" of each factor.

One can try to explain why this simpler system was not used by Archimedes. It might be due to the syntactical peculiarities of Ancient Greek in which the numbers "myriad-myriad" and "myriad-myriad-myriad" were written $\mu\upsilon\rho\iota\alpha\sigma$ $\mu\upsilon\rho\iota\alpha\delta\alpha\sigma$ and $\mu\upsilon\rho\iota\alpha\kappa\iota\sigma$ $\mu\upsilon\rho\iota\alpha\iota\sigma$ $\mu\upsilon\rho\iota\alpha\delta\epsilon\sigma\sigma\iota\nu$, respectively. This can be transliterated to "a myriad–times a myriad times a myriad." One sees that the Greek syntax does not make the repetition of the word "myriad" as clear as in present English usage.

As noted above, except for one case, Archimedes uses the law of exponents only in the form $10^a 10^6 = 10^{a+6}$. Clearly, Archimedes is picking the largest cube of a power of 10 smaller than 10^8, so that each multiplication will only increase numbers by a single order at most. Whether this is done because it makes the argument clearer, or because Archimedes is uncomfortable with his notation is a matter of debate.

As discussed above, to improve Archimedes' system, one should

1. Name cardinals instead of ranges of numbers.
2. Find a natural way to name large ordinals.
3. Make the law of exponents clearer.

One finds a basis for such a system in nomenclature of Diophantus [12, p. 47] who used the notation "first myriad" for $10,000$ and "second myriad" for $10,000^2$.

This system has all the above properties in that it names cardinals, each term in the sequence has a corresponding ordinal name, and the law of exponents is seen by the fact that the mth–myriad times the nth–myriad is the $(m+n)$th–myriad.

This process ends at the the myriadth myriad ($\mu\upsilon\rho\iota o\sigma\tau\alpha\ \mu\upsilon\rho\iota\alpha$) which can be called a "big myriad" ($\mu\epsilon\gamma\alpha\ \mu\upsilon\rho\iota\alpha$). If one lets $A = 10000$, then a big myriad is A^A which will be denoted as B. Similarly, a "bigger myriad" ($\mu\epsilon\iota\zeta\omega\nu\ \mu\upsilon\rho\iota\alpha$) would be the big myriadth big myriad ($\mu\epsilon\gamma\alpha\ \mu\upsilon\rho\iota o\sigma\tau\alpha\ \mu\epsilon\gamma\alpha\ \mu\upsilon\rho\iota\alpha$) which would be B^B, denoted by Γ. A "biggest myriad" would be the "bigger myriadth bigger myriad" ($\mu\epsilon\iota\zeta\omega\nu\ \mu\upsilon\rho\iota o\sigma\tau\alpha$) denoted by Δ and equal to Γ^Γ. Finally, one would have the "biggest myriadth biggest myriad" which would be Δ^Δ. This last number is larger than $10^{10^{10^{10^{10}}}}$ which is much larger than Archimedes' numbers, yet expressible in similar notation.

As with Archimedes, one can go further by letting A be the first period, B the second period, etc., so that the biggest myriadth period would be a number larger that $10^{10^{\cdot^{\cdot^{\cdot^{10}}}}}$ where there are $10^{10^{10^{10}}}$ terms in the exponential.

References

[1] Archimedes, *The Works of Archimedes,* edited in modern notation with introductory chapters by T.L. Heath, Dover, New York, 1953. Reprinted (translation only) in *Great Books of the Western World, Vol. 11,* R.M. Hutchins, editor, Encyclopaedia Britannica, Inc., Chicago, 1952. Translation of *The Sand Reckoner* reprinted in J.R. Newman, *World of Mathematics, Vol. 1,* Simon and Schuster, New York, 1956, 420–431.

[2] Archimedes, *Opera Omnia,* with commentary by Eutocius, edited by I.L. Heiberg and additional corrections by E.S. Stamatis, B.G. Teubner, Stuttgart, 1972.

[3] Archimède, *Oeuvres, 4 vol.,* texte établi et traduit par C. Mugler, Les Belles Lettres, Paris, 1970–71.

[4] Aristarchus, *On the sizes and distances of the sun and moon*, in [17], 351–411.

[5] Aristotle, *De Caelo,* Loeb??

[6] Aristophanes, *Archanians,* translated by B.B. Rogers, in *Aristophanes, Vol. 1,* Loeb Classical Library **??**, Harvard University Press, Cambridge, MA, 19??.

[7] Aristophanes, *Archanians,* translated with introduction and notes by Jeffrey Henderson, Focus Classical Library, Newburyport, MA, 1992.

[8] I. Asimov, *How did we Find out that the Earth is Round?,* Walker & Co., New York 1972.

[9] C.B. Boyer, *A History of Mathematics,* John Wiley & Sons, New York 1991.

[10] R. Calinger, *Classic of Mathematics,* Prentice Hall, Englewood Cliffs, NJ 1995.

[11] Cleomedes, *De Motu Circulari Corporum Caelestium, duo libri,* H. Ziegler, editor, Teubner, Leipzig 1891.

[12] T.L. Heath, *Diophantus of Alexandria, a study in the history of Greek algebra,* Cambridge University Press, Cambridge, 1910.

[13] H. Eves, *An Introduction to the History of Mathematics,* Saunders College Publishing, 1984.

[14] D.H. Fowler, *The Mathematics of Plato's Academy: a New Reconstruction* , Clarendon Press, Oxford 1987.

[15] O. Gingerich, *The Eye of Heaven: Ptolemy, Copernicus, and Kepler,* American Institute of Physics, New York 1993.

[16] R.M. Green, *Spherical Astronomy,* Cambridge University Press, Cambridge 1985.

[17] T.L. Heath, *Aristarchus of Samos, the Ancient Copernicus,* Dover, New York 1981.

[18] T.L. Heath, *Ancient Astronomy,* Dover, New York 1981.

[19] T.L. Heath, *A History of Greek Mathematics, Vol. 2,* Dover, New York 1981.

[20] S. Hornblower and A. Spawforth, *The Oxford Classical Dictionary, third edition,* Oxford University Press, New York 1996.

[21] Homer, *The Iliad,* translated by A.T. Murray, Loeb Classical Library **170 171**, Harvard University Press, Cambridge, MA, 1988.

[22] D.E. Knuth, Supernatural Numbers, in *The Mathematical Gardner,* edited by D.A. Klarner, Wadsworth, Belmont, CA, 1981, 310–325

[23] A. Koyré, *From the Closed World to the Infinite Universe,* Johns Hopkins University Press, Baltimore 1957.

[24] K. Lasky, *The Librarian who Measured the Earth,* Little, Brown & Co., Boston 1994.

[25] A. Lejeune, *La dioptre d'Archimède,* Annales de la Société Scientifique de Bruxelles, Sér. I **61** (1947), 27–47.

[26] Liddell and Scott, *Greek–English Lexicon,* ??.

[27] O. Neugebauer, *Astronomy and History: Selected Essays,* Springer–Verlag, New York 1983.

[28] Pindar, *Olympian Odes, Pythian Odes*, edited and translated by W.H. Race, Loeb Classical Library **56**, Harvard University Press, Cambridge, MA, 1997.

[29] Plato, *Timaeus,* translated by ??, Loeb Classical Library **234**, Harvard University Press, Cambridge MA, 1914.

[30] C. Ptolemy, *The Almagest,* translated by R.C. Taliaferro, in "Ptolemy, Copernicus, Kepler", Encyclopaedia Britannica, Chicago 1952.

[31] C. Ptolemy, *The Almagest,* translated and annotated by G.J. Toomer, Springer Verlag, New York 1984.

[32] A.E. Shapiro, Archimedes's measurement of the sun's apparent diameter, *Journal for the History of Astronomy,* **6** (1975), 75–83.

[33] J. Strong, *Strong's Exhaustive Concordance of the Bible,* World Publishing, Grand Rapids, MI, 1986.

[34] I. Thomas, *Greek Mathematical Works II, Aristarchus to Pappus of Alexandria,* Loeb Classical Library **362**, Harvard University Press 1980.

[35] I. Vardi, Archimedes' Cattle Problem, *Am. Math. Monthly,* to appear.