# Reading seminar: Hopf algebras

András Kornai
SZTAKI Institute for Computer Science
and
Dept of Algebra, Budapest Institute of Technology

## 0 Background

These notes are intended as background for the fall seminar where we will read three papers: Marcolli, Chomsky, and Berwick (2023), Marcolli, Berwick, and Chomsky (2023), and Nemecek (2023). The first two papers recast Merge, a central operation in Chomsky's Minimalist Grammar (Chomsky, 1995), in terms of Hopf algebras, while the third one argues that all building blocks of transformer models can be expressed via Hopf algebras. If both of these claims make sense, we are witnessing a convergence between the theoretical and the computational approaches to syntax not seen since the 1960s.

We will review the papers with great care. Over the summer we have built the definition of Hopf Algebras from the ground up, and (depending on need) we may repeat this material in the fall.

Chomsky and others have been wrestling with Merge from the get-go, with several alternative formulations proposed over the years (see e.g. Collins and Stabler, 2016). But experience shows that very complex definitions can have significant bugs, see e.g. Svenonius (1958), so a great deal of care is required, and mechanization of proofs seems advisable. Over the summer two experts gave their introduction to Minimalism, Avery Andrews and Diego Krivochen, and students are advised to watch the recorded talks.

We will also probe some larger questions that go beyond bug-bashing. First, (natural language) syntax already has some highly abstract formulations, including **Sadrzadeh:2015**; Lambek, 1958; Lambek, 2004; Clark, Coecke, and Sadrzadeh, 2013, and others. What are the natural language constructs that these formulations address, or fail to adress? Many practicing syntacticians feel that the gap between the abstract theories and the concrete linguistic facts needs to be bridged.

Second, since Hopf algebras are canonically built over vector spaces, investing some effort in relating them to the current use of vector spaces in computational

linguistics would be a good idea.

Third, algebra and coalgebra are important tools not just for investigating natural languages but also for theoretical informatics in general. Where Hopf algebras fit into this larger picture needs to be better understood.

Fourth, language acquisition is a major issue. We are interested not just in some abstract representational target which can be leveraged into efficient computational blocks, but also in the question of how these can be acquired from the data, In general, structures with finite combinatorics are incredibly hard to acquire Angluin, 1980; Angluin, 1981; Angluin, 1987. Obviously, the quantum physics people who also use Hopf algebras have no interest in this issue, but for linguistics acquisition has been considerd central since Chomsky, 1965.

**Format** We will have weekly hybrid meetings, if you wish to participate, please sign up by clicking this link and filling the questionnaire. We had an org meeting on the 11th of September, weekly meetings will be held every Monday at 10PM (Budapest time) on the 18th and afterwards likely alternating between a 6:30PM and a 10PM slot.

There is a **course webpage** at https://nessie.ilab.sztaki.hu/∼kornai/2023/Hopf, a page also reachable as kornai.com → 2023 → Hopf. This has a complete record of what we did over the summer, and will continue in the fall.

If you intend to participate, please also sign up for the course **slack** https://join.slack.com/t/hopfalgebraseminar/shared_invite/zt-23e5ugl43-qfkDbD_uEyRmZCcwgaDj_g

Over the summer we began to cover several background papers, including McCulloch and Pitts, 1943; Little, 1974 and Smolensky, 1990 (see lecture by Gerald Penn). The videos have about 80% overlap between the 2PM and the 11PM sessions, but there were important things in the discussions that may make it worth your while to view both.

Check the slides for **homeworks** in three categories: beginner, intermediate, and advanced. Some of these are quite hard, and some require theorem proving knowledge. These provide an opportunity to gauge your level of understanding, evaluation will depend on course activity, including giving presentations.

We also had an introduction to Nemecek (2023) by Michael Bukatin, and he identified two key readings we will also discuss, Elhage et al. (2021) and Nanda et al. (2023)

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