Hopf algebra reading seminar

András Kornai

Aug 2 2023 2PM CET

Organization

- 2pm zoom https://us02web.zoom.us/j/84045659802?pwd=L3grbWtqREE4OE
- 11pm zoom https://us02web.zoom.us/j/89203668566?pwd=M1dRL2ozOWxBT
- Slack https://join.slack.com/t/slack-qyx1689/shared_invite/zt-1xppi4d00-WnJhAvg_ThoSBOw9xH7yIw
- Course webpage https://nessie.ilab.sztaki.hu/~kornai/2023/Hopf
  Also reachable as kornai.com → 2023 → Hopf
- Attendance sheet https://docs.google.com/spreadsheets/d/17cK-cl3_xdbo73_kHWCIAvwgkd-G6qz44J4D6tyFfAc/edit?usp=sharing

Plan for today: HAs piece by piece

1. Tensor products (Blanka Kövér)
2. Convolution
3. Antipodes

Tensor products

Paper:
https://nessie.ilab.sztaki.hu/∼kornai/2023/Hopf/Resources/gowers_tens

Why introduce tensor products?

- Let V, W, X be vector spaces over $\mathbb{R}$. $f : V \times W \to X$ is *bilinear* if it is linear in each variable when we fix the other variable, i.e. $f(av + bv', w) = af(v, w) + bf(v', w)$ and $f(v, cw + dw') = cf(v, w) + df(v, w')$
- We already know linear maps correspond to matrices. Is there a similar way to naturally encode *bilinear* maps using a few numbers?
- Yes! If v_i and w_j are bases of V and W respectively, then because of the bilinearity of f, knowing $f(v_i, w_j)$ for all pairs of basis vectors determines f. If $\dim V = p$, $\dim W = q$ and $\dim X = r$, then pqr numbers are enough to describe f.

When is f completely determined?

- We do *not* necessarily need to know all of the $f(v_i, w_j)$'s
  - Ex. $V = W = \mathbb{R}^2$, $X = \mathbb{R}$ with the usual basis e_1, e_2
  - $f(e_1, e_1)$, $f(e_2, e_2)$, $f(e_1 + e_2, e_1 + e_2)$, $f(e_1 + e_2, e_1 + 2e_2)$ determines f, since we can solve a system of linear equations and express $f(e_1, e_2)$ and $f(e_2, e_1)$
  - $f(e_1, e_1)$, $f(e_2, e_2)$, $f(e_1 + e_2, e_1 + e_2)$, $f(e_1 - e_2, e_1 - e_2)$ does *not* determine f
- We want *something like a 'basis' of pairs* (v, w). The basis of $V \times W$ does not work (see the usual basis in $\mathbb{R}^2$)

When is f completely determined?

Try special cases to get a feel for the problem

- $V = W = \mathbb{R}$: Suppose $f(a, b)$ is given. Then $f(x, y) = \frac{xy}{ab} f(a, b)$ if $ab \neq 0$
- $V = W = \mathbb{R}^2$: Suppose $f(s, t)$, $f(u, v)$, $f(w, x)$, $f(y, z)$ is given. We can take (s, t) and (u, v) to be such that s and u form a basis of V, and t and v form a basis of W
- If $w = as + bu$, $x = ct + dv$, $y = es + gu$ and $z = ht + kv$, then $f(w, x) = acf(s, t) + adf(s, v) + bcf(u, t) + bdf(u, v)$ and $f(y, z) = ehf(s, t) + ekf(s, v) + ghf(u, t) + gkf(u, v)$ by bilinearity
- If $adgh \neq bcek$, we get a unique solution for $f(s, v)$ and $f(u, t)$, and hence f is determined
- If $adgh = bcek$, then for any f bilinear, $ekf(w, x) - adf(y, z) = (ekac - adeh)f(s, t) + (ekbd - adgk)f(u, v)$ is *automatically satisfied*. This looks like linear dependence, but isn't quite that

Converting the problem into linear algebra (1st way)

- $B := \{\text{bilinear maps defined on } V \times W\}$. Regard (u, v) as a *function* on B: $(u, v)(f) := f(u, v)$. Notation: $[u, v]$
- B is too big to be a set! Gowers shows that it is enough to consider bilinear maps to $\mathbb{R}$. So redefine $B := \{\text{bilinear maps } V \times W \to \mathbb{R}\}$
- Using this notation, our previous equation becomes $ek[w, x] - ad[y, z] = (ekac - adeh)[s, t] + (ekbd - adgk)[u, v]$. This is genuine linear dependence in the vector space of functions from B to $\mathbb{R}$
- Why was this all useful? Now we have
    - $\{(v_i, w_i)\}$ fixes all bilinear maps iff $[v, w] \in \langle\{[v_i, w_i]\}\rangle$ $(v, w) \in V \times W$
    - $\{(v_i, w_i)\}$ contains no redundancies iff the functions $[v_i, w_i]$ are linearly independent

Converting the problem into linear algebra (2nd way)

- Facts
  1. $[v, w + w'] - [v, w] - [v, w'] = 0$
  2. $[v + v', w] - [v, w] - [v', w] = 0$
  3. $[av, w] - a[v, w] = 0$
  4. $[v, aw] - a[v, w] = 0$
- Proposition: A linear combination of functions of the form $[v, w]$ is zero if and only if it is generated by functions of the form $[av, w] - a[v, w]$, $[v, aw] - a[v, w]$, $[v, w + w'] - [v, w] - [v, w']$ and $[v + v', w] - [v, w] - [v', w]$.
- Proof: $\Leftarrow$ is trivial, $\Rightarrow$ on the next slide

Converting the problem into linear algebra (2nd way)

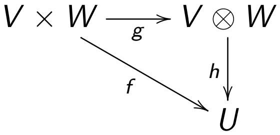
- $Z := \{\text{linear combinations of } [[v, w]]\}$ where $[[v, w]]$ is a meaningless symbol
- $E :=$ the subspace of Z generated by vectors of the form $[[av, w]] - a[[v, w]]$, $[[v, w + w']] - [[v, w]] - [[v, w']]$, $[[v, aw]] - a[[v, w]]$, $[[v + v', w]] - [[v, w]] - [[v', w]]$
- We want everything in E to 'be zero' in some sense, so we take the quotient space Z/E. This gives a trivial proof of the proposition:
  - Suppose $a_1[v_1, w_1] + \ldots + a_n[v_n, w_n]$ is not a linear combination of expressions of such forms
  - Then of course $z := a_1[[v_1, w_1]] + \ldots + a_n[[v_n, w_n]]$ is not a linear combination of vectors of the form above
  - Equivalently, $z \notin E$, or $z + E \neq 0 \in Z/E$
  - Then for $f(v, w) = [[v, w]] + E$, $a_1 f(v_1, w_1) + \ldots + a_n f(v_n, w_n) = z + E \neq 0$ as we wanted

How to think about tensor products

- $v \otimes w$: an alternative notation for $[v, w]$ or for $[[v, w]] + E$
- an element of $V \otimes W$: a *linear combination* of elements $v \otimes w$. NOT all elements are pure tensors!
- Gowers' advice: do NOT pay undue attention to the construction! When working with an equation involving tensors, do not worry about what the objects mean, instead use the following fact: $a_1 v_1 \otimes w_1 + \ldots a_n v_n \otimes w_n = 0$ iff $a_1 f(v_1, w_1) + \ldots + a_n f(v_n, w_n) = 0$ for all f bilinear
- If V and W are finite dimensional with given bases, we can also think of tensors as matrices: if $v = (a_1, \ldots, a_m)$ and $w = (b_1, \ldots, b_n)$, then $v \otimes w$ is the matrix $((A_{ij}))$ with $A_{ij} = a_i b_j$
  - Advantage: easier to visualize
  - Disadvantage: relies on a particular choice of basis

Universal property

- Take $g : V \times W \to V \otimes W$ which takes (v, w) to $v \otimes w$
- g is bilinear (check this using our previous fact), and g is 'arbitrary' in the following sense: $a_1 g(v_1, w_1) + \ldots + a_n g(v_n, w_n) = 0$ iff $a_1 f(v_1, w_1) + \ldots + a_n f(v_n, w_n) = 0$ for all f bilinear. We say the tensor product has a *universal property*
- If $f : V \times W \to U$ is bilinear, then we can define $h : V \otimes W \to U$ by sending $v \otimes w$ to $f(v, w)$ (extend linearly). Then $h \circ g = f$ and h is the only such linear map, so we say f *factors uniquely through g*



Convolution

- Classic setup: $f, g \ \mathbb{R} \to \mathbb{R}$ functions with compact support, $(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$
- $*$ is obviously a bilinear function, $*$ is associative
- Modern generalization: let G be a group with locally compact Hausdorff topology (group multiplication and inverse are continuous!). G has Haar measure μ on the Borel subsets (σ-algebra generated by the open sets) of G. With functions $f, g : G \to \mathbb{C}$ we can define $f * g(t)$ as $\int_G f(s)g(s^{-1}t)d\mu(s)$
- Related to Möbius inversion formula (see lecture 20 of Ardila) and to Lagrange inversion formula (not discussed by FA)
- Call H, when viewed as an algebra H^a, and call it H^c when viewed as a coalgebra
- If $f, g : H^c \to H^a$ are linear functions, their *convolution* $f * g$ is defined as the composition

$$H^c \xrightarrow{\Delta} H^c \otimes H^c \xrightarrow{f \otimes g} H^a \otimes H^a \xrightarrow{m} H^a$$

Antipode

- First, let's see that $*$ as defined above is bilinear, associative
- Now consider Id: $H \to H$ (obviously a linear function)
- The *antipode* S is the inverse of Id for convolution:
  $S * Id = Id * S = \eta \circ \epsilon$
- By def, HAs have antipodes, but we can construct near-HAs that meet all other requirements but have no antipode
- Antipode is unique if it exists
- Let's build a diagram, work out some examples