Hopf algebra reading seminar

András Kornai

July 19 2023 2PM CET

Organization

- 2pm zoom
  https://us02web.zoom.us/j/84045659802?pwd=L3grbWtqREE4OE
- 11pm zoom
  https://us02web.zoom.us/j/89203668566?pwd=M1dRL2ozOWxBT
- Slack https://join.slack.com/t/slack-qyx1689/shared_invite/zt-1xppi4d00-WnJhAvg_ThoSBOw9xH7yIw
- Course webpage
  https://nessie.ilab.sztaki.hu/~kornai/2023/Hopf
  Also reachable as kornai.com → 2023 → Hopf
- Attendance sheet
  https://docs.google.com/spreadsheets/d/17cK-cl3_xdbo73_kHWCIAvwgkd-G6qz44J4D6tyFfAc/edit?usp=sharing

Plan for today: HAs piece by piece

1. Algebras
2. Coalgebras
3. Tensors
4. This still leaves *antipodes*

Algebra

- A is a *vector space* over some field $\mathbb{K}$ (or more generally, *module* over some ring R) (so it comes with *scalar multiplication* and *addition*)
- Has a *product* $\cdot$ that is associative (not necessarily commutative) with a unit (some people talk about *unital* algebras to stress this point)
- Product *distributes* over the addition, making it a *ring*, the ring addition is the same as the module addition
- Isomorphic copy of $\mathbb{K}$ is built into the center of A (if we do this over a ring R, we only require homomorphic copy)

Canonical examples: $n \times n$ matrices over a ring/field, polynomial rings, group rings, any ring as a $\mathbb{Z}$-module

Group ring is an algebra

- Take any group G (can be finite or infinite, commutative or not), create finite sums $\sum \lambda_i g_i$ where $\lambda_i \in \mathbb{K}, g_i \in G$
- This will be a vector space over $\mathbb{K}$ with termwise addition, group elements form a basis, $\dim(A) = |G|$
- Multiply $\sum \lambda_i g_i$ with $\sum \mu_j g_j$ as $\sum_{i,j} \lambda_i \mu_j g_i g_j$ (every term by every term)
- Check that $\{\lambda e | \lambda \in \mathbb{K}\}$ is isomorphic to K
- Check A-unit, associativity, distributivity
- group product $\neq$ tensor product! (we don't even have $\otimes$ yet, and it will be strange)
- Homework: prove that any ring R is a $\mathbb{Z}$-module

Grading

- Modeled on the polynomial case
- let $\deg(p)$ be defined as usual
- We have $\deg(pq) = \deg(p) + \deg(q)$
- Whenever we have such a $\deg : A \to \mathbb{Z}$ we call A a *graded algebra*
- These can be naturally presented as a sequence $\{A_i | i \in \mathbb{N}\}$ where A_i contains the degree i monomials
- How about multivariate polynomials? What goes in the A_i?

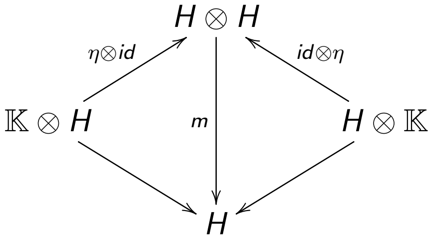
Coalgebras

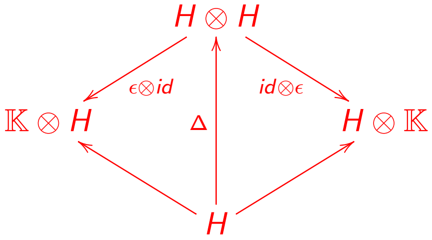
- We will first restate the algebraic requirements (associativity, unit) that are imposed on algebras as diagrams
- We reverse the diagrams 'dualize'
- Coalgebras are structures that satisfy the reverse diagrams
- We will provide examples
- Duality is a huge powerful thing!

Diagrams: unit

We use $\otimes$ for tensor product: $m : H \otimes H \to H$ with unit $\eta : \mathbb{K} \to H$

We use Δ for coproduct: $\Delta : H \to H \otimes H$ with counit $\epsilon : H \to \mathbb{K}$





Diagrams: associativity

We use $\otimes$ for tensor product: $m : H \otimes H \to H$ with unit $\eta : \mathbb{K} \to H$

We use Δ for coproduct: $\Delta : H \to H \otimes H$ with counit $\varepsilon : H \to \mathbb{K}$

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{m \otimes id} & H \otimes H \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ H \otimes H & \xrightarrow{m} & H \end{array}$$

$$\begin{array}{ccc} H \otimes H \otimes H & \xleftarrow[\Delta \otimes id]{} & H \otimes H \\ \uparrow{\scriptstyle id \otimes \Delta} & & \uparrow{\scriptstyle \Delta} \\ H \otimes H & \xleftarrow[\Delta]{} & H \end{array}$$

Examples of coalgebras (Ardila Lecture 5)

- Built on the algebra of sets $\mathbb{K}S$
- Let $\Delta : s \mapsto x \otimes s$, and $\epsilon : s \mapsto 1$
- Verify this is a coalgebra
- Built on poset intervals: let $I = \{z | x \leq z \leq y\}$ the base, we work in $\mathbb{K}I$
- Let $\Delta : [x, y] \mapsto \sum_{x \leq z \leq y} [x, z] \otimes [z, y]$
- Let $\epsilon([x, y]) = 1$ if $x = y$, 0 otherwise
- Verify this is a coalgebra

Algebras and coalgebras in a more general setting

- First we generalize 'algebra' from vector spaces to any category $\mathcal{C}$ (incl. Set, where objects are sets, arrows are functions) endowed with an endofunctor F
- If F is a functor (it maps objects to objects, arrows to arrows, preserves composition and identity) from $\mathcal{C}$ to $\mathcal{C}$, an *algebra* for F is defined by a set X and a function from $F(X)$ to X
- Example: fix $F(X)$ to be $X + 1$ (discrete union of X and a one-member set denoted *) this amounts to endowing each set with a distinguished element $f(*)$ and a unary operator s.
- A key example is $X = \mathbb{N}, s(*) = 0, s(n) = n + 1$
- This is the (unique) *initial object* among F-algebras
- 'unique' always means 'up to isomorphism'

Initial and final objects in a category

- An initial object is the 'smallest' object in a category: for I initial and X in $\mathcal{C}$ there is only one arrow from I to X
- A final (terminal) object is the 'largest': for T terminal and any X there is only one arrow from X to T
- An object can be both initial and final, these are called *null* objects
- What is a *coalgebra* for the same F? This is given by the 'extended natural numbers' $\mathbb{N} \cup \{\infty\}$
- $\mathcal{C}$, $\mathcal{C}^{op}$, covariant, contravariant
- Generally, if A is an algebra for $\mathcal{C}$ with endofunctor F, A^* is a coalgebra means it's an algebra for $\mathcal{C}^{op}$ endowed with F^{op}

Tensors

- This is the abstract view – the concrete view (specifically tied to finite dimensional vector spaces) will be presented by Blanka Kövér two weeks from now
- Our category is composed of the modules (vector spaces) over the same fixed ring (field) R. The arrows are the multilinear mappings among these
- The *tensor product* $V \otimes W$ of two modules V, W (which don't have to have the same dimension) is an object $V \otimes W$ endowed with (incoming) arrow $\phi : V \times W \to V \otimes W$ such that for every module Z and incoming bilinear mapping $f : V \times W \to Z$ there exists a unique *linear* mapping $\tilde{f}$ such factors through ϕ