Hopf algebra reading seminar

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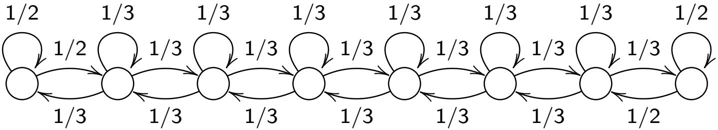
Organization

- 2pm zoom
  https://us02web.zoom.us/j/84045659802?pwd=L3grbWtqREE4OE
- 11pm zoom
  https://us02web.zoom.us/j/89203668566?pwd=M1dRL2ozOWxBT
- Slack https://join.slack.com/t/slack-qyx1689/shared_invite/zt-1xppi4d00-WnJhAvg_ThoSBOw9xH7yIw
- Course webpage
  https://nessie.ilab.sztaki.hu/∼kornai/2023/Hopf
  Also reachable as kornai.com → 2023 → Hopf
- Attendance sheet
  https://docs.google.com/spreadsheets/d/17cK-cl3_xdbo73_kHWCIAvwgkd-G6qz44J4D6tyFfAc/edit?usp=sharing

Plan for today

1. Self-assessment discussion with TP focus
2. Prediction versus explanation

State diagrams: the easy part



$$
\begin{bmatrix}
1/2 & 1/2 & & & & & & \\
1/3 & 1/3 & 1/3 & & & & & \\
 & 1/3 & 1/3 & 1/3 & & & & \\
 & & 1/3 & 1/3 & 1/3 & & & \\
 & & & 1/3 & 1/3 & 1/3 & & \\
 & & & & 1/3 & 1/3 & 1/3 & \\
 & & & & & 1/3 & 1/3 & 1/3 \\
 & & & & & & 1/2 & 1/2
\end{bmatrix}
$$

See Blanka Kövér's notebook at Resources/EigvalBlankaKover.ipynb

State diagrams: the hard part

- We are interested in the *second largest* eigenvalue in the general case (n states).
- What do we know? By Gershgorin's theorem, all eigenvalues are within the union of the disk of radius $1/2$ around the point $(1/2,0)$ and the disk of radius $2/3$ around $(1/3,0)$. In particular, all eigenvalues $\lambda \neq 1$ satisfy $|\lambda| < 1$
- *Can we do something simpler first?*
- Yes, we can *symmetrize* How do we recognize the pattern?
- This will eliminate the pesky first and last rows (which pollute the second and next-to-last columns as well)
- So this could be a win, but we are losing something at the same time so it's a gambit

Let's compute the eigenvalues of T_n:

$$\begin{bmatrix} 1/3 & 1/3 & & & \\ 1/3 & 1/3 & 1/3 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/3 & 1/3 & 1/3 \\ & & & 1/3 & 1/3 \end{bmatrix}$$

- What do we know? By symmetry, all eigenvalues are real, by Gershgorin's theorem, all eigenvalues are within the disk of radius $2/3$ around $(1/3{,}0)$ so they are all in the closed interval $[-1/3, 1]$ Can we do better?
- Let λ be an eigenvalue, $[x_1, \ldots, x_n]$ a corresponding eigenvector.
- For $1 < i < n$ we have $\frac{1}{3}x_{i-1} + (\frac{1}{3} - \lambda)x_i + \frac{1}{3}x_{i+1} = 0$
- Standard trick: Let $x_0 = x_{n+1} = 0$ making the above true for $i = 1$ and $i = n$ as well
- *What does this remind you of?*

Change of frame

- This is a Fibonacci-like recurrence, let's do generating functions!
- Let $P(z) = \sum_{i=0}^{n} x_i z^i$, consider $\frac{1}{3}P(z) + (\frac{1}{3} - \lambda)P(z)z + \frac{1}{3}P(z)z^2$
- This gives $P(z) = \frac{z(\frac{1}{3}x_1 + \frac{1}{3}x_n z)}{\frac{1}{3} + (\frac{1}{3} - \lambda)z + \frac{1}{3}z^2}$
- Our new frame is analysis. We have a bag of tricks here, of we (based on the Fibonacci analogy) we deploy which one?
- How do we even begin to search? We apply some standard heuristics

Pólya (1945)

HOW TO SOLVE IT

UNDERSTANDING THE PROBLEM

First.

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition?

Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second.

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

Pólya cont'd

If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

CARRYING OUT THE PLAN

Third.
Carry out your plan.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

LOOKING BACK

Fourth.
Examine the solution obtained.

Can you *check the result*? Can you check the argument?
Can you derive the result differently? Can you see it at a glance?
Can you use the result, or the method, for some other problem?

Change of mental direction

- Instead of trying to solve for λ, we try to solve for the x_i
- Notice that in $\frac{z(\frac{1}{3}x_1+\frac{1}{3}x_n z)}{\frac{1}{3}+(\frac{1}{3}-\lambda)z+\frac{1}{3}z^2}$ we have a quadratic denominator. What to do?
- Use *partial fractions*. "The concept was discovered independently in 1702 by both Johann Bernoulli and Gottfried Leibniz" (WP, citing Grosholz (2000))
- Let the roots of the denominator be r_1 and r_2. We have $x_i = Br_1^i + Cr_2^i$ for some constants B, C
- The quadratic formula is no help, what to do?

From 'standard trick' to foundation stone

- We have $B + C = 0$ from $x_0 = 0$ and $Br_1^{n+1} + Cr_2^{n+1} = 0$ from $x_{n+1} = 0$
- We also have from the Vieta formulas $r_1 + r_2 = 3(\lambda - \frac{1}{3})$ and $r_1 r2 = \frac{1}{3} / \frac{1}{3}$
- Now for something clever:
- $B(\frac{r_1}{r_2})^{n+1} + C = 0$ but also $B + C = 0$ so $\frac{r_1}{r_2}$ is an $n + 1$st root of unity.
- What was a 'standard trick' (letting $x_0 = x_{n+1} = 0$) became key to the solution
- From Vieta we have $\lambda = \frac{1}{3} + \frac{2}{3} \cos \frac{k\pi}{n+1}$

What is missing?

- Going back from T_n to A_n is not trivial. Clearly, determinant is a continuous function of the entries, so coefficients of the characteristic polynomial also are, and roots are continuous function of the coefficients
- So in principle, this is doable, but in practice still very hard. Can a human do it? Can a TP system?
- By and large, the method described for T_n should work for A_n except it is precisely the first and the last equation which are different, which will affect the recursion
- Self-assessment (beginner): compute the eigenvector v_i corresponding to λ_i
- Self-assessment (intermediate/advanced): study Kato's *Perturbation Theory for Linear Operators* and see what you can apply here

Stanley Peters

In science there is a familiar distinction between prediction and explanation. Consider, for example, solar eclipses. Tradition has it that Thales made the first prediction of a solar eclipse in the 6th century BCE. (That may not be accurate because there's no evidence that the fact that solar eclipses are simply occultation of the sun by the moon was known until a century later.) However, the predictability of solar eclipses wasn't understood until the 17th century CE, when Newton explained them by figuring out how the relative motions of the sun, earth, and moon related to their mutual gravitational attractions. The first eclipse predicted by relying on this explanation occurred about three decades later when Halley did the calculations and predicted a 1715 eclipse in England.

Or consider Brownian motion: the movement of particles suspended in a fluid. While any given occurrence of Brownian motion is describable only statistically (e.g. as a Gaussian Markov process with independent increments, etc.), its properties are explained by causal features like the energy of the fluid exerting forces randomly on the particles, the resistance of the fluid to movement of the particles, the particles' masses, etc.

Prediction v. Explanation

My [S.P.] question is to what extent do LLMs trained by deep learning aim to explain properties of language such well-formedness, dependency structure, logical consistency of meaning, etc.? Are they aiming simply to predict 'properly formed' results of language generation by humans – analogous to the Saros Cycle of solar eclipses? Or do they aim to explain why humans generate the language we do – analogous to Halley's calculation from Newton's laws together with constants for the mass and distances of the sun, earth, and moon?
If the latter, where in them will we find (a) the explanatory hypotheses and (b) the 'constants' for what the LLM intends to communicate?

AK: questions to SP

1. Where do you put Panini's grammar on the simulation/explanation scale? I myself consider it highly explanatory, but you may disagree strongly. Where do you put Varro, the Arab grammatical tradition, and American structuralist descriptive grammars of indian languages? How about tagmemic grammars (for many languages these are the only grammars we have). In a similar vein, where do you put Stockwell, Schachter, and Partee 1968? How about Huddleston and Pullum 2002, Culicover and Jackendoff 2005? Are there any grammars you consider explanatory?

2. Do you see the prediction/explanation distinction as polar opposites? Where do you put similation models, e.g. a contemporary planetarium, the https://en.wikipedia.org/wiki/Antikythera_mechanism or the https://www.researchgate.net/publication/3048623_A_Digital_Orrery?

3. Finally, the independence of these two notions. Can there be explanation without prediction? I think this is what Chomsky aims at with the core/periphery distinction, let's do gonzo explanation without bothering with the details (I may be too cynical here).