Hopf algebra seminar

In Search of an Antipode

Blanka Kövér

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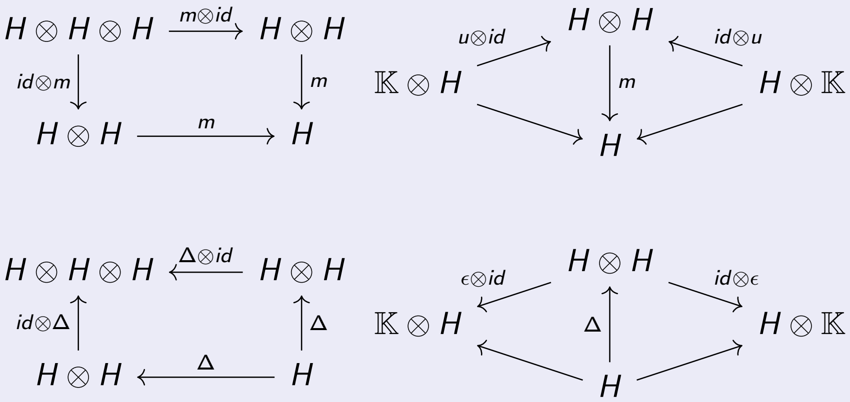
Quick recap

Definition

A *bialgebra* H is a $\mathbb{K}$-vector space such that the following diagrams commute:



Example

The group ring $\mathbb{K}G = \left\{ \sum_{i=1}^{n} \lambda_i g_i : \lambda_i \in \mathbb{K}, g_i \in G \right\}$.

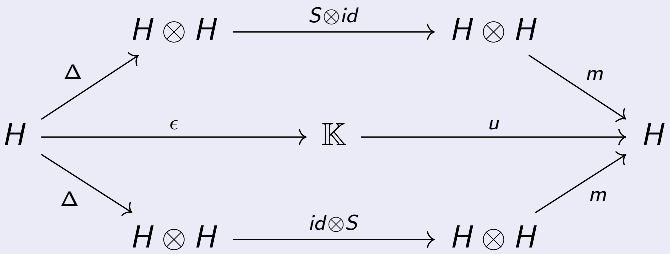
Here we have

$$\begin{array}{cc} m(g \otimes h) = gh & u(\lambda) = \lambda \cdot 1_G \\ \Delta(g) = g \otimes g & \epsilon(g) = 1_{\mathbb{K}} \end{array}$$

(extend linearly).

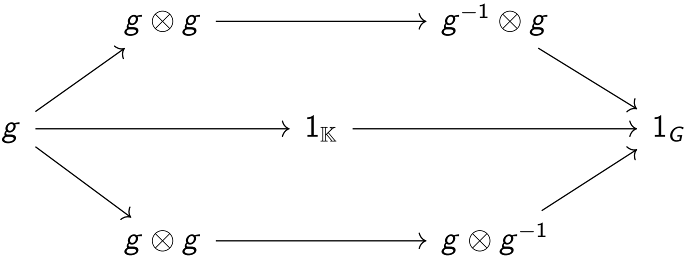
Definition

An *antipode* on a bialgebra H is a linear map $S : H \to H$ such that the following diagram commutes:



Idea: S is kind of like an inverse.

In the case of group ring, $S(g) = g^{-1}$ indeed makes it a Hopf algebra:



$S : H \to H$ is an antipode for H iff $S * id = u\epsilon = id * S$. S is unique if it exists.
Reminder: the *convolution* of f and g is the composition

$$f * g = m \circ (f \otimes g) \circ \Delta$$

Bialgebras without an antipode

If G is a monoid but not a group, then $\mathbb{K}G$ is not a Hopf algebra.

Example

S is a set, $G = \mathcal{P}(S)$ its power set. Let $A \cdot B = A \cap B$ for $A, B \in G$. Then G is a monoid with $1_G = S$ and no inverses.

Note from András: these structures are very important because the syntactic monoids of most formal languages are not groups.

More generally, if our bialgebra H contains elements that are 'not invertible', then H cannot be a Hopf algebra.

Definition

An element $q \in H$ is *grouplike* if $\Delta(q) = q \otimes q$ and $\epsilon(q) = 1_{\mathbb{K}}$.

The set of grouplike elements forms a monoid under multiplication. If H is a Hopf algebra with antipode S, then S acts like an inverse operation for grouplike elements, making this monoid into a group. So if there is a noninvertible grouplike element, H cannot admit an antipode.

Example

For those with a taste for algebraic topology: see **Beauvais** for a description of a topological example, the p-primary *Dyer-Lashof algebra*.

Note: this is not the only way for a bialgebra not to have an antipode, see **Radford**.

In search of an antipode

Takeaway: we don't actually need an antipode, but in the cases we consider, its existence is automatically guaranteed.

Definition

A bialgebra H is *graded* if

- $H = \bigoplus_{n=0}^{\infty} H_n$
- $H_i H_j \subseteq H_{i+j}$ for all $i, j \geq 0$
- $\Delta H_n \subseteq \bigoplus_{i+j=n} (H_i \otimes H_j)$
- $\epsilon H_n = 0$ for all $n \geq 1$

H is *connected* if $H_0 \cong \mathbb{K}$.

Example

$H = \mathbb{K}[x]$ with $m(x^i \otimes x^j) = x^{i+j}$, $\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j$ and

$H_n = \mathbb{K}\{x^n\}$

Example

$H = \mathbb{K}\{$isomorphism classes of finite graphs$\}$ with

- multiplication: disjoint union
- comultiplication: $\Delta(G) = \sum_{S \subseteq V} G|_S \otimes G|_{V-S}$, i.e. partitions of the graph

$H_n = \mathbb{K}\{$isomorphism classes of finite graphs with n vertices$\}$

Theorem (Takeuchi, 1971)

A graded, connected bialgebra has an antipode, explicitly given by the formula

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

where $\pi = id - u\epsilon$.

Convention: $m^0 = \Delta^0 = id$, $m^{-1} = u$, $\Delta^{-1} = \epsilon$.
That is to say, "often the antipode comes for free" (Ardila).

Fact

$S(h_d)$ is a finite sum for any $h_d \in H_d$.

- Let $n > d$.
- $\Delta^{n-1}(H_d) \subseteq \bigoplus_{i_1+\ldots+i_n=d} (H_{i_1} \otimes \ldots \otimes H_{i_n})$
- Since $n > d$, $i_j = 0$ for some j.

Fact

$\pi(H_0) = 0$.

This comes from the assumption that H is connected (in fact they are equivalent). Since $H_0 \cong \mathbb{K}$, $u\epsilon|_{H_0} = id|_{H_0}$.

- Hence $m^{n-1}\pi^{\otimes n}\Delta^{n-1}(H_d) = 0$ if $n > d$.
- So $S(h_d)$ is indeed finite.

Fact

$m^{n-1}\pi^{\otimes n}\Delta^{n-1} = \pi * \ldots * \pi = \pi^{*n}$

(feel free to check this!)
Then

$$S * id = \left(\sum_{n\geq 0}(-1)^n \pi^{*n}\right) * (\pi + u\epsilon)$$
$$= \sum_{n\geq 0}(-1)^n \pi^{*(n+1)} + \sum_{n\geq 0}(-1)^n \pi^{*n} = \pi^{*0} = u\epsilon$$

Similarly for $id * S$. □

Antipodes in MCB and MBC

- MCB: gives an (inductive) definition for the antipode, and then never uses it again.
- MBC: "Thus, in the following we will focus only on the bialgebra structure and not discuss the antipode map explicitly."
- The bialgebra of (planar) binary rooted trees is graded by the number of leaves.
- Moral of the story: we can forget about the antipode (for now) while knowing that our structures are indeed Hopf algebras.

- The Ardila lectures, as always
- *Why graded bi-algebras have antipodes*, Secret Blogging Seminar, https://sbseminar.wordpress.com/2011/07/07/why-graded-bi-algebras-have-antipodes/
- J. Beauvais-Feisthauer, Y. Patel, A. Salch, *Milnor-Moore Theorems for Bialgebras in Characteristic Zero*, Journal of Algebra, **623** (2023) 234–268.
- David E. Radford, *On bialgebras which are simple Hopf modules*, Proc. Amer. Math. Soc., **80(4)** (1980) 563–568.