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ON BIALGEBRAS WHICH ARE SIMPLE HOPF MODULES

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ABSTRACT. This paper gives a module characterization of commutative or cocommutative Hopf algebras over a field.

0. Introduction. Let A be a bialgebra over a field k. Then A has a natural left A-Hopf module structure, and if A is a Hopf algebra, an easy calculation with the antipode shows that A is a simple Hopf module. We show that a commutative or cocommutative bialgebra over a field k which is a simple left Hopf module is a Hopf algebra. From this result we derive a module-theoretic characterization of commutative or cocommutative bialgebras over a field k which are Hopf algebras; namely such a bialgebra is a Hopf algebra if and only if all left A-Hopf modules are free (or (0)).

Generally a commutative or cocommutative bialgebra A over a field k has a unique maximal subcoalgebra $A_{(I)}$ which is a Hopf algebra. In both cases $A_{(I)}$ can be described in terms of grouplike elements–the basic results of this paper are derivatives of elementary observations concerning grouplikes in certain bialgebras.

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1. Preliminaries. In this section we show that any bialgebra A over a field k has a unique subcoalgebra $A_{(I)}$ maximal among the subcoalgebras D such that the inclusion $i_D \in \mathrm{Hom}(D, A)$ has an inverse in the convolution algebra. We will see that $A_{(I)}$ is characterized by its simple subcoalgebras.

For a coalgebra C over k recall that the wedge product $U \wedge V$ of subspaces $U, V \subseteq C$ is defined by $U \wedge V = \Delta^{-1}(U \otimes C + C \otimes V)$. The wedge product of subcoalgebras is a subcoalgebra.

LEMMA 1. *Let C be a coalgebra over a field k, and suppose $E, D', D'' \subseteq C$ are subcoalgebras, E simple.*

(a) *If $E \subseteq \sum D$, where D runs over a family of subcoalgebras of C, then $E \subseteq D$ for some D.*

(b) *If $E \subseteq D' \wedge D''$ then $E \subseteq D'$ or $E \subseteq D''$.*

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(c) *If A is a bialgebra and $E \subseteq D'D''$, then $E \subseteq E'E''$ where $E' \subseteq D'$ and $E'' \subseteq D''$ are simple subcoalgebras.*

(d) *If $f\colon C' \to C$ is a surjective coalgebra map, then $E \subseteq f(E')$ for some simple subcoalgebra $E' \subseteq C'$.*

PROOF. (a) is [S, Proposition 8.0.3.a]. To show (b) note that $\Delta E \subseteq D' \otimes C + C \otimes D''$ means $(E \otimes E) \cap (D' \otimes C + C \otimes D'') \neq (0)$. By (a) if U is a simple subcoalgebra of this intersection, then $U \subseteq D' \otimes C$ or $U \subseteq C \otimes D''$, so $E \subseteq D'$ or $E \subseteq D''$. To show (d) observe that $C_0 \subseteq f(C_0')$ by [**HR**, 2.3.9]. Thus writing $C_0' = \Sigma E'$ as the (direct) sum of the simple subcoalgebras of C', we have $E \subseteq \Sigma f(E')$, and hence (d) follows by (a). To show (c), first observe that $E \subseteq m(U)$ for some simple subcoalgebra $U \subseteq D' \otimes D''$ by (d), where $m\colon D' \otimes D'' \to D'D''$ is multiplication. By [**HR**, 2.3.13] $(D' \otimes D'')_0 \subseteq D_0' \otimes D_0''$, so $U \subseteq E' \otimes E''$ for simple subcoalgebras $E' \subseteq D'$ and $E'' \subseteq D''$ by (a). Thus $E \subseteq E'E''$. Q.E.D.

For a coalgebra C and an algebra A over k recall that the unity of the convolution algebra $\mathrm{Hom}(C, A)$ is given by $\boldsymbol{\varepsilon}(c) = \varepsilon(c)1_A$ and the product is $f * g(c) = \Sigma f(c_{(1)})g(c_{(2)})$ for $c \in C$ and $f, g \in \mathrm{Hom}(C, A)$. The following is a refinement of [T, Lemma 14].

LEMMA 2. *Let C be a coalgebra and A be an algebra over a field k, and let $x \in \mathrm{Hom}(C, A)$.*

(a) *If $x \equiv 0$ on C_0 then $X \mapsto x$ determines an algebra map $k[[X]] \mapsto \mathrm{Hom}(C, A)$. (Thus x is invertible if $x \equiv \boldsymbol{\varepsilon}$ on C_0.)*

(b) *x is left (resp. right) invertible if and only if $x|_E \in \mathrm{Hom}(E, A)$ is left (resp. right) invertible for all simple subcoalebras $E \subseteq C$. (Hence x is invertible if and only if $x|_E$ is invertible for all simple subcoalgebras $E \subseteq C$.)*

PROOF. (a) The C_n's form a filtration of C. Thus if $x \equiv 0$ on C_0, then $x^{n+1} \equiv 0$ on C_n for $n \geqslant 0$ by induction, and therefore $\Sigma_{n=0}^{\infty} \alpha_n x^n$ is meaningful for all $\alpha_0, \alpha_1, \ldots \in k$. That $X \mapsto x$ extends to an algebra map is easy to check. If $x \equiv \boldsymbol{\varepsilon}$ on C_0, we have just shown that $X \mapsto \boldsymbol{\varepsilon} - x$ determines an algebra map, so $x = \boldsymbol{\varepsilon} - (\boldsymbol{\varepsilon} - x)$ is invertible since $1 - X \in k[[X]]$ is.

(b) If x is left invertible, then $x|_D$ is for any subcoalgebra $D \subseteq C$. On the other hand, if $x|_E$ is left invertible for all simple subcoalgebras $E \subseteq C$, then $x|_{C_0}$ has a left inverse $f \in \mathrm{Hom}(C_0, A)$ since C_0 is a direct sum of simples. Let $F\colon C \to A$ be a linear extension of f. Then $F * x \equiv \boldsymbol{\varepsilon}$ on C_0 which means $F * x$ is invertible by (a), hence x is left invertible. The rest easily follows. Q.E.D.

PROPOSITION 1. *Let C be a coalgebra and A be an algebra over a field k, and let $f \in \mathrm{Hom}(C, A)$.*

(a) *Let $L_{(f)} \subseteq C$ (resp. $R_{(f)} \subseteq C$) be the sum of all subcoalgebras $D \subseteq C$ such that $f|_D$ is left (resp. right) invertible. Then $f|_{L_{(f)}}$ is left invertible and $f|_{R_{(f)}}$ is right invertible.*

(b) *$C_{(f)} = L_{(f)} \cap R_{(f)}$ is the sum of all subcoalgebras $D \subseteq C$ such that $f|_D$ is invertible, and $f|_{C_{(f)}}$ is invertible.*

(c) *$L_{(f)}$ and $R_{(f)}$ (hence $C_{(f)}$) are closed under wedging.*

(d) *If* Ω *is a field extension of* k *then* $L_{(f)} \otimes_k \Omega \subseteq L_{(f \otimes I_\Omega)}$ *and* $R_{(f)} \otimes_k \Omega \subseteq R_{(f \otimes I_\Omega)}$ (*hence* $C_{(f)} \otimes_k \Omega \subseteq C_{(f \otimes I_\Omega)}$).

(e) *If* A *is a bialgebra and* f *is a coalgebra homomorphism, then* $f|_{C_{(f)}}^{-1}$ *is a coalgebra antihomomorphism.*

PROOF. (a) follows by Lemmas 2(b) and 1(a). (b) follows directly from (a). That $L_{(f)}$ and $R_{(f)}$ are closed under wedging, or equivalently, $L_{(f)} \wedge L_{(f)} \subseteq L_{(f)}$ and $R_{(f)} \wedge R_{(f)} \subseteq R_{(f)}$, follows by Lemmas 2(b) and 1(b). (d) is straightforward. The proof of [**HS**, 1.5.2.(6)] generalizes to a proof of (e). Q.E.D.

Let A be a bialgebra over k and let $L_{(I)}$, $R_{(I)}$ and $A_{(I)}$ denote the subcoalgebras of $C = A$ described in Proposition 1 for the identity map I of A. One should note that $G(A_{(I)})$ consists of all invertible $g \in G(A)$ and is therefore a group. (For a coalgebra C over k recall that $g \in C$ is grouplike if $g \neq 0$ and $\Delta g = g \otimes g$, and that $G(C)$ denotes the set of grouplike elements of C.) By Proposition 1(e) $s = (I|_{A_{(I)}})^{-1}$ is a coalgebra antihomomorphism. Observe that $s(g) = g^{-1}$ for $g \in G(A_{(I)})$.

Suppose that U and V are vector spaces over k. For a field extension Ω of k $(U \otimes_k \Omega) \otimes_k (V \otimes_k \Omega) \simeq (U \otimes_k V) \otimes_k \Omega((u \otimes \alpha) \otimes (v \otimes \beta) \mapsto (u \otimes v) \otimes \alpha\beta)$ is an isomorphism of Ω-spaces. The Galois group $G(\Omega \setminus k)$ acts on $U \otimes_k \Omega$ as k-automorphisms by the rule $\sigma \cdot (u \otimes \alpha) = u \otimes \sigma\alpha$. (Thus $G(\Omega \setminus k)$ acts as k-algebra automorphisms if U is an algebra.)

LEMMA 3. *Let* C *be a coalgebra over a field* k *and suppose* Ω *is a field extension of* k. *Then* $G(C \otimes_k \Omega)$ *is a* $G(\Omega \setminus k)$*-module, which is cyclic if* C *is cocommutative and simple and* Ω *is an algebraic closure of* k.

PROOF. If $g = \sum_i c_i \otimes \alpha_i \in C \otimes_k \Omega$, then $\Delta g = g \otimes g$ if and only if $\sum_i \Delta c_i \otimes \alpha_i = \sum_{i,j} c_i \otimes c_j \otimes \alpha_i\alpha_j$. From this it follows that $G(C \otimes_k \Omega)$ is a $G(\Omega \setminus k)$-module. Now assume C is cocommutative and simple, and Ω is an algebraic closure of k. The isomorphism $C \otimes_k \Omega \simeq \mathrm{Hom}_k(C^*, \Omega)$ $(c \otimes \alpha(c^*) = c^*(c)\alpha)$ restricts to an identification of $G(\Omega \setminus k)$-modules $G(C \otimes_k \Omega) \simeq \mathrm{Alg}_k(C^*, \Omega)$. Since C^* is a finite-dimensional field extension of k and Ω is an algebraic closure of k, given $\tau, \tau' \in \mathrm{Alg}_k(C^*, \Omega)$ there exists a $\sigma \in G(\Omega \setminus k)$ such that $\tau' = \sigma \circ \tau$, i.e. $\mathrm{Alg}_k(C^*, \Omega)$ is cyclic. Q.E.D.

Let V be a left C-module with basis $v_1, \ldots, v_n$. Define $e_{ij} \in C$ $(1 \leqslant i, j \leqslant n)$ by $\omega(v_i) = \sum_{j=1}^n e_{ij} \otimes v_j$. Then $\Delta e_{ij} = \sum_k e_{ik} \otimes e_{kj}$ and $\varepsilon(e_{ij}) = \delta_{ij}$ follow from the comodule axioms. If C is a commutative bialgebra then the determinant $d = \det(e_{ij})$ of $(e_{ij}) \in M(n, C)$ is grouplike and does not depend on the choice of basis. Here we set $d_V = d$.

Let $C(n, k)$ be the coalgebra over k with basis of symbols e_{ij} $(1 \leqslant i, j \leqslant n)$ with structure defined as above. Let $V \subseteq C(n, k)$ have basis $e_{11}, \ldots, e_{n1}$ and let $S(C(n, k))$ be the free commutative bialgebra on $C(n, k)$. Then $\mathfrak{U}_n(k) = S(C(n, k))[d_V^{-1}]$ is a Hopf algebra since it represents the affine group scheme $\mathrm{GL}_n(\ )$ over k.

2. The main results. Here we examine the role of grouplikes in certain bialgebras.

PROPOSITION 2. *Let A be a bialgebra over a field k which is commutative or has cocommutative coradical. Then $A_{(I)}$ is a Hopf algebra.*

PROOF. Assume A is commutative. We will use Nichols' result [N] that a bialgebra quotient of a commutative Hopf algebra is a Hopf algebra. First let $V \subseteq A_{(I)}$ be a simple subcoalgebra and choose $e_{ij} \in A$ $(1 \leqslant i, j \leqslant n)$ for V as indicated above. Since V is a coalgebra, the e_{ij}'s span V. From the equations $\sum_k e_{ik}s(e_{kj}) = \delta_{ij}1$ we conclude that $(e_{ij}) \in M(n, A)$ is invertible, hence d_V is invertible. Clearly $B_V = (V)[d_V^{-1}]$ is a bialgebra quotient of $\mathfrak{U}_n(k)$, so B_V is a Hopf algebra. In particular $s(V) \subseteq A_{(I)}$. For simple subcoalgebras $E', E'' \subseteq A_{(I)}$ we apply Nichols' result again to multiplication $B_{E'} \otimes B_{E''} \to B_{E'}B_{E''}$ to deduce $E'E'' \subseteq A_{(I)}$. Thus $A_{(I)}$ is a Hopf algebra by Lemmas 2(b) and 1(c).

Now assume A_0 is cocommutative and Ω is an algebraic closure of k. Then $A \otimes_k \Omega$ is pointed. $D = A_{(I)} + s(A_{(I)})$ is a subcoalgebra of A by Proposition 1(e). Note $G(s(A_{(I)}) \otimes_k \Omega) = G(s \otimes I(A_{(I)} \otimes_k \Omega)) = G(A_{(I)} \otimes_k \Omega)^{-1}$ by Lemma 1(d), so $G(D \otimes_k \Omega) = G(D \otimes_k \Omega)^{-1}$ by part (a) of the same lemma. Thus by Lemma 1(c) the grouplikes of $(D) \otimes_k \Omega = (D \otimes_k \Omega)$ form a group. Therefore $(D) \otimes_k \Omega$ is a Hopf algebra by Lemma 2(b), and this means (D) is a Hopf algebra also. By definition $A_{(I)} = (D)$. Q.E.D.

Using Lemma 3 we have as a corollary to the proof:

COROLLARY 1. *Let A be a bialgebra over a field k, and suppose $C \subseteq A$ is a simple subcoalgebra.*

(a) *If A is commutative, then $C \subseteq A_{(I)}$ if and only if d_C is invertible in A.*

(b) *If A_0 is cocommutative and Ω is an algebraic closure of k, then $C \subseteq A_{(I)}$ if and only if some $g \in G(C \otimes_k \Omega)$ is invertible in $A \otimes_k \Omega$.*

$A_{(I)}$ need not be a Hopf algebra in general.

EXAMPLE 1. Let $C = C(n, k) \oplus \mathbf{C}(n, k)$ and $T(C)$ be the free bialgebra on C. The ideal $I \subseteq T(C)$ generated by the relations described in $M(n, T(C))$ by $(e_{ij})(\mathbf{e}_{ij})^t = I = (\mathbf{e}_{ij})^t(e_{ij})$ is a bi-ideal. By the method of §1 of [B] one can show that $C \subseteq A \equiv T(C)/I$ (so $C(n, k) \subseteq A_{(I)}$) and that $(\mathbf{e}_{ij}) \in M(n, A)$ is not invertible (so $s(C(n, k)) = \mathbf{C}(n, k) \not\subseteq A_{(I)}$) for $n \geqslant 2$.

PROPOSITION 3. *Let A be a bialgebra over a field k and suppose $C, C' \subseteq A$ are nonzero subcoalgebras such that $CC' \subseteq A_{(I)} \supseteq C'C$. Then $C, C' \subseteq A_{(I)}$.*

PROOF. Assume $CC' \subseteq A_{(I)}$ and choose $a' \in C'$ such that $\varepsilon(a') = 1$. Then $t \in \operatorname{Hom}(C, A)$ defined by $t(c) = \sum a'_{(1)}s(ca'_{(2)})$ is a right inverse for $I|_C$, so $C \subseteq R_{(I)}$. Likewise $C' \subseteq L_{(I)}$. Q.E.D.

Let A be a bialgebra with left antipode s (a left inverse of $I \in \operatorname{Hom}(A, A)$). Such an A is a *left Hopf algebra*. Suppose $V \subseteq A$ is a left Hopf submodule (i.e. $\Delta V \subseteq A \otimes V$ and $AV \subseteq V$). For $v \in V$ the calculation $\varepsilon(v)1 = \sum s(v_{(1)})v_{(2)} \in AV = V$ shows $V = (0)$ or $V = A$, so A is a simple left A-Hopf module.

THEOREM 1. *Let A be a bialgebra over a field k which is commutative or has cocommutative coradical. Then the following are equivalent.*

(a) *A is a Hopf algebra.*

(b) *A is a simple left A-Hopf module.*

(c) *If $C \subseteq A$ is a simple subcoalgebra then $AC = A$.*

PROOF. We need only show (c) $\Rightarrow$ (a). Assume (c) holds and let $C \subseteq A$ be simple. By Lemma 2(b) we need only show $C \subseteq A_{(I)}$. If A is commutative, d_C is invertible, so $C \subseteq A_{(I)}$ by Corollary 1(a). Suppose A_0 is cocommutative and Ω is an algebraic closure of k. Then $AC = A$ means $1 \in C'C$ for some simple C' by Lemma 1(c), so $1 = g'g$ where $g' \in G(C' \otimes_k \Omega)$ and $g \in G(C \otimes_k \Omega)$ by the same result. Now replacing C by C' we see $1 = h''h'$ for some $h' \in G(C' \otimes \Omega)$. P $g' = \sigma \cdot h'$ for some $\sigma \in G(\Omega \setminus k)$. Thus the calculation $1 = (\sigma \cdot h'')(\sigma \cdot h')$ shows that g' is invertible, so g is also, and $C \subseteq A_{(I)}$ by Corollary 1(b). Q.E.D.

A bialgebra which is a simple left Hopf module may not be a Hopf algebra (there exist left Hopf algebras which are not Hopf algebras [GNT]).

As a consequence of Lemma 2(b) and Corollary 1(a) a commutative bialgebra is a Hopf algebra if and only if all grouplikes are invertible [T, Corollary 69]. Generally this is not the case.

EXAMPLE 2. Let C be a coalgebra over a field k and let $T(C)$ be the free bialgebra on C. Then $T(C)^n = C \otimes \cdots \otimes C$ (n-times) has the tensor product coalgebra structure ($n > 1$). From the general coalgebra fact that

$$G(C_1 \otimes \cdots \otimes C_n) = G(C_1) \times \cdots \times G(C_n)$$

and Lemma 1(a) we conclude that $G(T(C)) = \{1\}$ if $G(C) = \varnothing$. For example let $k = \mathbf{R}$ and $C = \mathbf{C}^*$. Then $T(C)$ is a cocommutative bialgebra, but is not a Hopf algebra, since $T(C) \otimes_{\mathbf{R}} \mathbf{C} \simeq T(C \otimes_{\mathbf{R}} \mathbf{C})$ is the free monoid on $G(C \otimes_{\mathbf{R}} \mathbf{C})$.

EXAMPLE 3. Let C be a coalgebra which is also an algebra (possibly without unity), and suppose Δ, ε are multiplicative. The coalgebra structure of C extends (uniquely) to a bialgebra structure on the algebra $A = k \cdot 1 + C$ obtained by adjoining a unity to C. A is not a Hopf algebra, since C is a sub-Hopf module, unless $C = (0)$. For finite-dimensional examples where $G(A) = \{1\}$ let $C = C(n, k)$ and $e_{ij}e_{kl} = \delta_{ij}e_{kl}$.

Our last result gives a module-theoretic characterization of commutative or cocommutative Hopf algebras.

THEOREM 2. *Let A be a bialgebra over a field k which is commutative or has cocommutative coradical. Then A is a Hopf algebra if and only if all left A-Hopf modules are free* (*or* (0)).

PROOF. If A is any Hopf algebra over a field, then all left Hopf modules are free (or (0)) by [S, Theorem 4.1.1]. Conversely, suppose A satisfies this condition and let $C \subseteq A$ be a simple subcoalgebra. By Theorem 1 we must show $AC = A$, or $A/AC = (0)$. Since AC and A/AC are Hopf modules, AC is free and A/AC is free or (0). Since $\ker \varepsilon$ is a codimension 1 ideal, any two bases of a free A-module M

have the same cardinality $r(M)$. If A/AC is free, then $1 = r(A) = r(AC) + r(A/AC)$, a contradiction, so $A/AC = (0)$. Q.E.D.

"Free" cannot be replaced by "projective" in the preceding theorem since there are semisimple bialgebras which are not Hopf algebras. We close with a general construction.

EXAMPLE 4. Let $\mathfrak{A}$ be an associative algebra (with unity) over k with an algebra map $\delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ satisfying $I \otimes \delta \circ \delta = \delta \otimes I \circ \delta$. The direct sum of algebras $A = k \cdot d \oplus \mathfrak{A}$ $(d^2 = d \neq 0)$ has a bialgebra structure determined by $\Delta d = d \otimes d$ and $\Delta a = a \otimes d + d \otimes a + \delta a$ for $a \in \mathfrak{A}$. A is a Hopf algebra if and only if $\mathfrak{A} = (0)$. Let $\mathfrak{A}$ be any semisimple bialgebra.

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