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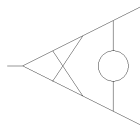
# An introduction to Hopf algebras of trees

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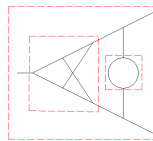
## Introduction

The Hopf algebra of rooted trees is introduced in Connes and Kreimer [1998], in the context of Quantum Field Theory. The considered problem is the following: an integral is attached to certain graphs, called the Feynman graphs, according to certain rules, called the Feynman rules. It turns out that these



**Fig. 1.** A Feynman graph

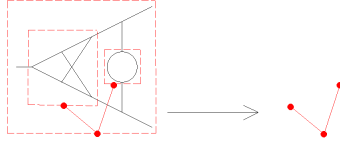
integrals are divergent, because of the presence of loops in Feynman graphs: each loop of the graph creates a subdivergence in the associated integral. The



**Fig. 2.** The subdivergences of the graph

Renormalization procedure (Collins [1984]) is used to give these integrals a sense, despite their divergences. In the Connes-Kreimer point of view, the Renormalization consists to associate to each Feynman graph a rooted (eventually decorated) tree representing the structure of the subdivergences of the

graph. After a regularization step, that is to say the introduction of a new



**Fig. 3.** Rooted tree associated to the graph

parameter  $h$ , the Feynman rules give now an algebra morphism  $\phi$  from the algebra of rooted trees  $\mathcal{H}_R$  to the algebra of formal meromorphic functions  $A = \mathbb{C}[[h]][h^{-1}]$ . As  $\mathcal{H}_R$  is a Hopf algebra,  $\phi$  becomes an element of the character group of  $\mathcal{H}_R$  with values in  $A$ . Now, the algebra  $A$  can be decomposed into  $A = A_- \oplus A_+$ , where  $A_+ = \mathbb{C}[[h]]$  and  $A_- = h^{-1}\mathbb{C}[[h^{-1}]]$ . The aim becomes the obtention of a Birkhoff decomposition of  $\phi$ , that is to say a decomposition of the form  $\phi = \phi_-^{-1} * \phi_+$ , where  $\phi_\epsilon$  is a character of  $\mathcal{H}_R$  such that  $\phi_\epsilon$  taken on a rooted tree is an element of  $A_\epsilon$  for  $\epsilon = +$  or  $-$ . Because  $\mathcal{H}_R$  is graded and connected, an inductive process allows to compute  $\phi_+$  and  $\phi_-$ , and Connes and Kreimer proved that the Renormalization consists to replace  $\phi$  by  $\phi_+$  when  $h$  goes to 0.

Our aim here is to introduce the Hopf algebra of rooted trees  $\mathcal{H}_R$  and its non commutative version  $\mathcal{H}_{PR}$ , as well as several algebraic properties of these Hopf algebras, including duality and non associative structures. We restrict ourselves to non decorated rooted trees, but we have to mention that all the results here exposed can be generalized in the decorated cases. The text is organized as follows: the first section deals with  $\mathcal{H}_R$ . We first introduce rooted trees and rooted forests, and show how admissible cuts give a coproduct on  $\mathcal{H}_R$ , making it a bialgebra. We then describe a gradation of  $\mathcal{H}_R$  and use it to prove the existence of an antipode, given by cuts. The universal property of  $\mathcal{H}_R$  is given, with several examples. We conclude with a description of the dual Hopf algebra  $\mathcal{H}_R^*$ , related to the Grossman-Larson Hopf algebra (Grossman and Larson [1990, 2005]).

In the second section, we proceed in the same way with a non commutative version. Replacing rooted trees by planar rooted trees, we construct the Hopf algebra  $\mathcal{H}_{PR}$ , and give its universal property. It is proved that  $\mathcal{H}_{PR}$  is isomorphic to its dual, which makes perhaps the main difference with the commutative case.

In the last section, we introduce extra algebraic structures on these Hopf algebras. The first one is a preLie structure on the Lie algebra of primitive elements of the dual Hopf algebra  $\mathcal{H}_R^*$ . This structure is used to construct two Hopf subalgebras of  $\mathcal{H}_R$ , namely the ladder subalgebra and the Faà di



*Example 2.* Here are the rooted forests of weight  $\leq 4$ :

$$1, \dots, \mathfrak{!}, \dots, \mathfrak{!}\cdot, \mathfrak{V}, \mathfrak{!}, \dots, \mathfrak{!}\cdot, \mathfrak{!}\mathfrak{!}, \mathfrak{V}\cdot, \mathfrak{!}\cdot, \mathfrak{V}, \mathfrak{V}, \mathfrak{!}, \mathfrak{!}\cdot.$$

In order to make  $\mathcal{H}_R$  a bialgebra, we now introduce the notion of cut of a tree  $t$ . We orient the edges of  $t$  upwards, from the root to the leaves. A *non total cut*  $c$  of a tree  $t$  is a choice of edges of  $t$ . Deleting the chosen edges, the cut makes  $t$  into a forest denoted by  $W^c(t)$ . The cut  $c$  is *admissible* if any oriented path in the tree meets at most one cut edge. For such a cut, the tree of  $W^c(t)$  which contains the root of  $t$  is denoted by  $R^c(t)$  and the product of the other trees of  $W^c(t)$  is denoted by  $P^c(t)$ . We also add the total cut, which is by convention an admissible cut such that  $R^c(t) = 1$  and  $P^c(t) = W^c(t) = t$ . The set of admissible cuts of  $t$  is denoted by  $\text{Adm}_*(t)$ . Note that the empty cut of  $t$  is admissible; we put  $\text{Adm}(t) = \text{Adm}_*(t) - \{\text{empty cut, total cut}\}$ .

*Example 3.* Let us consider the rooted tree  $t = \mathfrak{!}\mathfrak{V}$ . As it has 3 edges, it has  $2^3$  non total cuts.

cut $c$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{V}$	total
Admissible?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}\mathfrak{!}$	$\cdot\mathfrak{V}$	$\mathfrak{!}\cdot$	$\dots\mathfrak{!}$	$\mathfrak{!}\dots$	$\mathfrak{!}\dots$	$\dots$	$\mathfrak{!}\mathfrak{V}$
$R^c(t)$	$\mathfrak{!}\mathfrak{V}$	$\mathfrak{!}$	$\mathfrak{V}$	$\mathfrak{!}\cdot$	$\times$	$\cdot$	$\mathfrak{!}$	$\times$	1
$P^c(t)$	1	$\mathfrak{!}$	$\cdot$	$\cdot$	$\times$	$\mathfrak{!}\cdot$	$\dots$	$\times$	$\mathfrak{!}\mathfrak{V}$

The coproduct of  $\mathcal{H}_R$  is defined as the unique algebra morphism from  $\mathcal{H}_R$  to  $\mathcal{H}_R \otimes \mathcal{H}_R$  such that, for all rooted tree  $t \in \mathbf{T}$ :

$$\Delta(t) = \sum_{c \in \text{Adm}_*(t)} P^c(t) \otimes R^c(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in \text{Adm}(t)} P^c(t) \otimes R^c(t).$$

As  $\mathcal{H}_R$  is the free associative commutative unitary algebra generated by  $\mathbf{T}$ , this makes sense.

*Example 4.* Following example 3:

$$\Delta(\mathfrak{!}\mathfrak{V}) = \mathfrak{!}\mathfrak{V} \otimes 1 + 1 \otimes \mathfrak{!}\mathfrak{V} + \mathfrak{!} \otimes \mathfrak{!} + \cdot \otimes \mathfrak{V} + \cdot \otimes \mathfrak{!} + \mathfrak{!} \otimes \cdot + \dots \otimes \mathfrak{!}.$$

**Lemma 1.** We define  $B^+ : \mathcal{H}_R \rightarrow \mathcal{H}_R$  as the operator which associates to any rooted forest  $t_1 \dots t_n$ , the rooted tree obtained by grafting the roots of  $t_1, \dots, t_n$  on a common new root. Then, for all  $x \in \mathcal{H}_R$ :

$$\Delta \circ B^+(x) = B^+(x) \otimes 1 + (\text{Id} \otimes B^+) \circ \Delta(x).$$

For example,  $B^+(\cdot \uparrow) = \mathcal{V}$ .

*Proof.* We can restrict ourselves to  $x = t_1 \dots t_n \in \mathbf{F}$ . Let us consider a non total admissible cut  $c$  of  $t = B^+(x) \in \mathbf{T}$ . Then the restriction of  $c$  to  $t_i$  gives an admissible cut of  $t_i$ , eventually total if  $c$  cuts the edge from the root of  $t$  to the root of  $t_i$ . In the other sense, if  $c_1, \dots, c_n$  are admissible cuts of  $t_1, \dots, t_n$ , then there exists a unique non total admissible cut  $c$  of  $t$  such that the restriction of  $c$  to  $t_i$  gives  $c_i$  for all  $i$ . Moreover,  $R^c(t) = B^+(R^{c_1}(t_1) \dots R^{c_n}(t_n))$  and  $P^c(t) = P^{c_1}(t_1) \dots P^{c_n}(t_n)$ . Hence:

$$\begin{aligned} \Delta(t) &= t \otimes 1 + \sum_{c_i \in \text{Adm}_*(t_i), 1 \leq i \leq n} P^{c_1}(t_1) \dots P^{c_n}(t_n) \otimes B^+(R^{c_1}(t_1) \dots R^{c_n}(t_n)) \\ &= B^+(x) \otimes 1 + (Id \otimes B^+) \left( \prod_{i=1}^n \sum_{c_i \in \text{Adm}_*(t_i)} P^{c_i}(t_i) \otimes R^{c_i}(t_i) \right) \\ &= B^+(x) \otimes 1 + (Id \otimes B^+)(\Delta(t_1) \dots \Delta(t_n)) \\ &= B^+(x) \otimes 1 + (Id \otimes B^+) \circ \Delta(x). \quad \square \end{aligned}$$

**Theorem 1.** *With this coproduct,  $\mathcal{H}_R$  is a bialgebra. The counit of  $\mathcal{H}_R$  is given by:*

$$\varepsilon : \begin{cases} \mathcal{H}_R \longrightarrow K \\ F \in \mathbf{F} \longrightarrow \delta_{1,F}. \end{cases}$$

*Proof.* We have to prove three points:

1.  $\Delta$  is a morphism of algebras.
2.  $\varepsilon$  is a counit of  $\Delta$ .
3.  $\Delta$  is coassociative.

By definition of  $\Delta$ , the first point is obvious. Let us show the second point. For any  $t \in \mathbf{T}$ , if  $c \in \text{Adm}(t)$ , then both  $P^c(t)$  and  $R^c(t)$  are nonempty forests, so:

$$(\varepsilon \otimes Id) \circ \Delta(t) = \varepsilon(t)1 + \varepsilon(1)t + \sum_{c \in \text{Adm}(t)} \varepsilon(P^c(t))R^c(t) = t.$$

In the same way,  $(Id \otimes \varepsilon) \circ \Delta(t) = t$ . So  $Id$ ,  $(\varepsilon \otimes Id) \circ \Delta$  and  $(Id \otimes \varepsilon) \circ \Delta$  are three algebra endomorphisms of  $\mathcal{H}_R$  which coincide on  $\mathbf{T}$ . As  $\mathbf{T}$  generates  $\mathcal{H}_R$ , they are equal. So  $\varepsilon$  is a counit of  $\Delta$ .

Let us now give the proof of the coassociativity of  $\Delta$ . We consider:

$$A = \{x \in \mathcal{H}_R / (\Delta \otimes Id) \circ \Delta(x) = (Id \otimes \Delta) \circ \Delta(x)\}.$$

As  $(\Delta \otimes Id) \circ \Delta$  and  $(Id \otimes \Delta) \circ \Delta$  are two algebra morphisms from  $\mathcal{H}_R$  to  $\mathcal{H}_R \otimes \mathcal{H}_R \otimes \mathcal{H}_R$ ,  $A$  is a subalgebra. Let  $x \in A$ . Then:

$$\begin{aligned}
(\Delta \otimes Id) \circ \Delta(B^+(x)) &= \Delta(B^+(x)) \otimes 1 + (\Delta \otimes Id) \circ (Id \otimes B^+) \circ \Delta(x) \\
&= B^+(x) \otimes 1 \otimes 1 + (Id \otimes B^+) \circ \Delta(x) \otimes 1 \\
&\quad + (Id \otimes Id \otimes B^+) \circ (\Delta \otimes Id) \circ \Delta(x), \\
(Id \otimes \Delta) \circ \Delta(B^+(x)) &= B^+(x) \otimes 1 \otimes 1 + (Id \otimes \Delta \circ B^+) \circ \Delta(x) \\
&= B^+(x) \otimes 1 \otimes 1 + (Id \otimes B^+) \circ \Delta(x) \otimes 1 \\
&\quad + (Id \otimes Id \otimes B^+) \circ (Id \otimes \Delta) \circ \Delta(x).
\end{aligned}$$

As  $x \in A$ , these two elements coincide, so  $B^+(x) \in A$ :  $A$  is stable under  $B^+$ . Let us now show that any forest  $F \in \mathbf{F}$  belongs to  $A$  by induction on  $n = \text{weight}(F)$ . If  $n = 0$ , then  $F = 1 \in A$ . If  $n \geq 1$  and  $F$  is not a tree, then  $F = t_1 \dots t_k$ , with  $k \geq 2$ , and the induction hypothesis holds for  $t_1, \dots, t_k$ . As  $A$  is a subalgebra,  $F \in A$ . If  $n \geq 1$  and  $F \in \mathbf{T}$ , we can write  $F = B^+(G)$ , with  $G \in \mathbf{F}$ , and the induction hypothesis holds for  $G$ . As  $A$  is stable under  $B^+$ ,  $F \in A$ . As a conclusion,  $A = \mathcal{H}_R$ , so  $\Delta$  is coassociative.  $\square$

### 1.3 gradation of $\mathcal{H}_R$ and antipode

For all  $n \in \mathbb{N}$ , we put  $\mathcal{H}_R(n) = \text{Vect}(\mathbf{F}(n))$ . So  $\mathcal{H}_R = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_R(n)$ . Moreover:

1. For all  $i, j \in \mathbb{N}$ ,  $\mathcal{H}_R(i)\mathcal{H}_R(j) \subseteq \mathcal{H}_R(i+j)$ ,
2. For all  $n \in \mathbb{N}$ ,  $\Delta(\mathcal{H}_R(n)) \subseteq \sum_{k+l=n} \mathcal{H}_R(k) \otimes \mathcal{H}_R(l)$ .

In other terms,  $\mathcal{H}_R$  is a *graded bialgebra*. Note that  $\mathcal{H}_R(0)$  is reduced to the base field  $K$ : we shall say that  $\mathcal{H}_R$  is *connected*. The dimension of  $\mathcal{H}_R(n)$ , namely the number of rooted forests of weight  $n$ , can be inductively computed, as explained in Broadhurst and Kreimer [2000a]:

**Proposition 1.** *For all  $n \in \mathbb{N}$ , we put  $r_n = \dim_K(\mathcal{H}_R(n))$ . Then:*

$$\sum_{n=0}^{\infty} r_n h^n = \prod_{n=1}^{\infty} \frac{1}{(1-h^n)^{r_{n-1}}}.$$

The sequence  $(r_n)_{n \geq 0}$  is the sequence A000081 of Sloane.

*Example 5.*

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$r_n$	1	1	2	4	9	20	48	115	286	719	1842	4766	12486	32973	87811

The following lemma implies that  $\mathcal{H}_R$  has an antipode, so is a Hopf algebra:

**Lemma 2.** *Let  $A$  be a graded connected bialgebra. Then  $A$  has an antipode.*

*Proof. First step.* Let us prove that  $\varepsilon$  is zero on  $A(n)$  if  $n \geq 1$ . We assume that this is false and we take  $x \in A(n)$ ,  $n \geq 1$ , minimal, such that  $\varepsilon(x) \neq 0$ . As  $A$  is connected, we put:

$$\begin{cases} \Delta(x) = x_1 \otimes 1 + 1 \otimes x_2 + \sum x' \otimes x'', \\ \sum x' \otimes x'' \in \sum_{i=1}^{n-1} A(i) \otimes A(n-i). \end{cases}$$

By minimality of  $n$ ,  $A(i)$  and  $A(n-i)$  are subspaces of  $\text{Ker}(\varepsilon)$  if  $1 \leq i \leq n-1$ . So  $(\text{Id} \otimes \varepsilon) \circ \Delta(x) = x_1 + 1\varepsilon(x_2) = x$ . By homogeneity of  $x$ ,  $x_1 = x$  and  $\varepsilon(x_2) = 0$ . Symmetrically,  $x_2 = x$  and  $\varepsilon(x_1) = 0$ : this contradicts  $x = x_1$ . So  $\text{Ker}(\varepsilon) = \bigoplus_{n \geq 1} A(n)$ , and for any  $x \in A(n)$ , with  $n \geq 1$ :

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in \sum_{i=1}^{n-1} A(i) \otimes A(n-i).$$

*Second step.* We can define by induction on  $n$  the following application:

$$S_g : \begin{cases} A \longrightarrow A \\ 1 \longrightarrow 1, \\ x \in A(n) \longrightarrow -x - \sum S_g(x')x'', \end{cases}$$

putting  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ . Clearly,  $m \circ (S_g \otimes \text{Id}) \circ \Delta(x) = \varepsilon(x)1$  for all  $x \in A$ . So  $S_g$  is a left antipode of  $A$ . It is also possible to define a right antipode  $S_d$  of  $A$ . As the convolution product  $*$  is associative,  $S_d = (S_g * \text{Id}) * S_d = S_g * (\text{Id} * S_d) = S_g$ , so  $A$  has an antipode  $S = S_g = S_d$ .  $\square$

*Remark 1.* Applying this result to the opposite bialgebra  $A^{op}$ , we deduce that it has an antipode  $S'$ , so  $S$  is invertible, with inverse  $S^{-1} = S'$ .

We now describe the antipode  $S$  of  $\mathcal{H}_R$ . As  $\mathcal{H}_R$  is commutative, its antipode is an algebra morphism, so it is enough to give the antipode of elements of  $\mathbf{T}$ .

**Theorem 2.** *Let  $t \in \mathbf{T}$ . Then:*

$$S(t) = \sum_{c \text{ non total cut of } t} (-1)^{n_c+1} W^c(t),$$

where  $n_c$  is the number of cut edges in  $c$ .

*Proof.* Induction on the weight  $n$  of  $t$ . If  $n = 1$ , then  $t = \bullet$ ,  $\Delta(\bullet) = \bullet \otimes 1 + 1 \otimes \bullet$ , so  $S(\bullet) = -\bullet$  and the result is true. If  $n \geq 2$ :

$$S(t) = -t - \sum_{c \in \text{Adm}(t)} S(P^c(t))R^c(t).$$

Let us consider a non empty and non total cut  $c$  of  $t$ . There exists a unique admissible cut  $c'$  of  $t$  such that the tree of  $W^{c'}(t)$  which contains the root of  $t$  is  $R^{c'}(t)$ , and denoting  $P^{c'}(t) = t_1 \dots t_k$ , the restriction of  $c$  to  $t_i$  is a non total cut  $c_i$  of  $t_i$ . Moreover,  $W^c(t) = W^{c_1}(t_1) \dots W^{c_k}(t_k)R^{c'}(t)$  and  $n_c = k + n_{c_1} + \dots + n_{c_k}$ . Because  $S$  is an algebra morphism, by the induction hypothesis:

$$S(P^{c'}(t)) = \sum_{c_i \text{ non total cut of } t_i, 1 \leq i \leq k} (-1)^{n_{c_1} + \dots + n_{c_k} + k} W^{c_1}(t_1) \dots W^{c_k}(t_k).$$

Combining all these assertions:

$$S(t) = -t - \sum_{c \text{ non empty and non total}} (-1)^{n_c} W^c(t),$$

which implies the result.  $\square$

#### 1.4 Cartier-Quillen cohomology and universal property of $\mathcal{H}_R$

Let  $C$  be a coalgebra and  $(B, \delta_G, \delta_D)$  be a bicomodule over  $C$ . The Cartier-Quillen cohomology of  $C$  with coefficients in  $B$ , dual notion of the Hochschild cohomology of an algebra, is the cohomology of the complex defined by  $X_n = \text{Hom}_K(B, C^{\otimes n})$ , and coboundary  $b_n : X_n \rightarrow X_{n+1}$  given by:

$$b_n(L) = (Id \otimes L) \circ \delta_G + \sum_{i=1}^n (-1)^i \left( Id_C^{\otimes(i-1)} \otimes \Delta \otimes Id_C^{\otimes(n-i)} \right) \circ L \\ + (-1)^{n+1} (L \otimes Id) \circ \delta_D.$$

In particular, the 1-cocycles are linear applications  $L : B \rightarrow C$  satisfying the following property:

$$\Delta \circ L = (Id \otimes L) \circ \delta_G + (L \otimes Id) \circ \delta_D.$$

Let us choose a group-like element of  $C$ , which we denote by 1. Consider now the bicomodule  $(C, \Delta, \delta_D)$ , with  $\delta_D(x) = x \otimes 1$  for all  $x \in C$ . A 1-cocycle is a linear endomorphism  $L$  of  $C$  satisfying: for all  $x \in C$ ,

$$\Delta \circ L(x) = (Id \otimes L) \circ \Delta(x) + L(x) \otimes 1. \quad (1)$$

In particular, lemma 1 implies that  $B^+$  is a 1-cocycle of  $\mathcal{H}_R$ . Moreover,  $(\mathcal{H}_R, B^+)$  satisfies the following property:

**Theorem 3 (Universal property of  $\mathcal{H}_R$ ).** *Let  $A$  be a commutative algebra and let  $L : A \rightarrow A$  be a linear application.*

1. *There exists a unique algebra morphism  $\phi : \mathcal{H}_R \rightarrow A$ , such that  $\phi \circ B^+ = L \circ \phi$ .*



2. If  $A$  is a Hopf algebra and  $L$  satisfies (1), then  $\phi$  is a Hopf algebra morphism.

*Proof. Unicity.*  $\phi$  is entirely determined on  $\mathbf{F}$  by the following properties:

$$\begin{cases} \phi(1) = 1, \\ \phi(t_1 \dots t_n) = \phi(t_1) \dots \phi(t_n), \\ \phi(B^+(t_1 \dots t_n)) = L(\phi(t_1) \dots \phi(t_n)). \end{cases}$$

*Existence.* As  $A$  is commutative,  $\phi(t_1) \dots \phi(t_n)$  does not depend of the order of the  $t_i$ 's, so these formulas define a linear application  $\phi : \mathcal{H}_R \rightarrow A$ . The first and second formulas imply that  $\phi$  is an algebra morphism, and the third one that  $\phi \circ B^+ = L \circ \phi$ . Let us now suppose that  $L$  satisfies (1) and let us prove that  $\phi$  is a Hopf algebra morphism. We have to prove the two following points:

1.  $\varepsilon \circ \phi = \varepsilon$ .
2.  $\Delta \circ \phi = (\phi \otimes \phi) \circ \Delta$ .

First, for all  $x \in A$ :

$$L(x) = (\varepsilon \otimes Id) \circ \Delta \circ L(x) = \varepsilon \circ L(x)1 + (\varepsilon \otimes L) \circ \Delta(x) = \varepsilon \circ L(x)1 + L(x),$$

so  $\varepsilon \circ L = 0$ . Let  $t \in \mathbf{T}$ . We can write it as  $B^+(F)$ ,  $F \in \mathbf{F}$ . Then:

$$\varepsilon \circ \phi(t) = \varepsilon \circ \phi \circ B^+(F) = \varepsilon \circ L \circ \phi(F) = 0,$$

so  $\varepsilon \circ \phi$  and  $\varepsilon$  are two algebra morphisms from  $\mathcal{H}_R$  to  $K$  which coincide on  $\mathbf{T}$ : they are equal. This proves the first point.

Let us now prove the second point. We put:

$$X = \{x \in \mathcal{H}_R / \Delta \circ \phi(x) = (\phi \otimes \phi) \circ \Delta(x)\}.$$

As  $\Delta \circ \phi$  and  $(\phi \otimes \phi) \circ \Delta$  are two algebra morphisms,  $X$  is a subalgebra of  $\mathcal{H}_R$ . Let  $x \in X$ . Then:

$$\begin{aligned} \Delta \circ \phi \circ B^+(x) &= \Delta \circ L \circ \phi(x) \\ &= L \circ \phi(x) \otimes 1 + (Id \otimes L) \circ \Delta \circ \phi(x) \\ &= \phi \circ B^+(x) \otimes 1 + (Id \otimes L) \circ (\phi \otimes \phi) \circ \Delta(x) \\ &= \phi \circ B^+(x) \otimes 1 + (\phi \otimes \phi) \circ (Id \otimes B^+) \circ \Delta(x) \\ &= (\phi \otimes \phi) \circ \Delta(B^+(x)), \end{aligned}$$

so  $B^+(x) \in X$ . Then  $X$  is a subalgebra of  $\mathcal{H}_R$  stable under  $B^+$ : it is  $\mathcal{H}_R$ .  $\square$

*Remark 2.* The first point of theorem 3 proves that  $(\mathcal{H}_R, B^+)$  is an initial object in the category of commutative algebras with a linear application, as mentioned in Moerdijk [2001].

*Examples 6.*

1. Let  $A$  be a commutative Hopf algebra and let  $L$  be a 1-cocycle of  $A$  such that  $L(1) = 0$ . The Hopf algebra morphism induced by the universal property is given by  $\phi(x) = \varepsilon(x)1_A$  for all  $x \in \mathcal{H}_R$ .
2. We take  $A = K[X]$ , with the coproduct defined by  $\Delta(X) = X \otimes 1 + 1 \otimes X$ . The following application is a 1-cocycle of  $A$ :

$$L : \begin{cases} K[X] \longrightarrow K[X] \\ P(X) \longrightarrow \int_0^X P(t)dt \end{cases}$$

The Hopf algebra morphism induced by the universal property is given by  $\phi(F) = \frac{1}{F!} X^{\text{weight}(F)}$  for all  $F \in \mathbf{F}$ , where the combinatorial coefficient  $F!$  is inductively defined in Brouder [2004], Hoffman [2003] by:

$$\begin{cases} \bullet! = 1, \\ (t_1 \dots t_k)! = t_1! \dots t_k!, \\ B^+(F)! = F! \text{weight}(B^+(F)). \end{cases}$$

Other similar examples are given in Zhao [2004].

### 1.5 Dual Hopf algebra

We first expose some results and notations concerning the graded duality. Let  $A$  be a  $\mathbb{N}$ -graded vector space, such that the homogeneous components of  $A$  are finite-dimensional.

1. The graded dual  $A^*$  is  $\bigoplus_{n \in \mathbb{N}} A(n)^*$ . Note that  $A^*$  is also a graded space, and  $A^{**} \approx A$ .
2.  $A \otimes A$  is also a graded space, with  $(A \otimes A)(n) = \sum_{i=0}^n A(i) \otimes A(n-i)$  for all  $n \in \mathbb{N}$ . Moreover,  $(A \otimes A)^* \approx A^* \otimes A^*$ .
3. Let  $A$  and  $B$  be two graded spaces, with finite-dimensional homogeneous components, and  $F : A \longrightarrow B$ , homogeneous of a certain degree  $d$ , that is to say  $F(A(n)) \subseteq B(n+d)$  for all  $n \in \mathbb{N}$ . Then there exists a unique  $F^* : B^* \longrightarrow A^*$ , such that if  $f \in B^*$ ,  $F^*(f)(x) = f \circ F(x)$  for all  $x \in A$ . Moreover,  $F^*$  is homogeneous of degree  $-d$ .

All these results imply that if  $(A, m, \Delta)$  is a graded Hopf algebra, then its graded dual inherits also a graded Hopf algebra given by  $(A^*, \Delta^*, m^*)$ .

We now give a combinatorial description of the dual Hopf algebra  $\mathcal{H}_R^*$ . We shall need the following notions:

1. For all forest  $F \in \mathbf{F}$ ,  $s_F$  is the number of rooted forest automorphisms of  $F$ . These coefficients are inductively defined by:

$$\begin{cases} s_1 = 1, \\ s_{B^+(F)} = s_F, \\ s_{t_1^{\alpha_1} \dots t_k^{\alpha_k}} = s_{t_1}^{\alpha_1} \dots s_{t_k}^{\alpha_k} \alpha_1! \dots \alpha_k!, \end{cases}$$

if  $t_1, \dots, t_k$  are distinct elements of  $\mathbf{T}$ .

2. Let  $F = t_1 \dots t_n$  and  $G$  be two elements of  $\mathbf{F}$ . A grafting of  $F$  over  $G$  is a forest obtained by the following operations: for each  $i$ ,  $t_i$  is concatenated to  $G$ , or the root of  $t_i$  is grafted in a vertex of  $G$ . If  $F, G, H \in \mathbf{F}$ , the number of ways of grafting  $F$  on  $G$  to obtain  $H$  is denoted by  $n'(F, G; H)$ .

*Example 7.* For  $F = \bullet$  and  $G = \mathbb{V}$ , there are four graftings of  $F$  over  $G$ , which are  $\bullet \mathbb{V}$ ,  $\mathbb{V}$ ,  $\mathbb{V}$ ,  $\mathbb{V} = \mathbb{V}$ . In particular,  $n'(\bullet, \mathbb{V}; \mathbb{V}) = 2$ .

The following lemma is proved in Foissy [2002b], Hoffman [2003]:

**Lemma 3.** *For all forests  $F, G, H \in \mathbf{F}$ , we denote by  $n(F, G; H)$  the coefficient of  $F \otimes G$  in  $\Delta(H)$ . Then  $n'(F, G; H)_{s_H} = n(F, G; H)_{s_F s_G}$ .*

We now describe the Hopf algebra  $\mathcal{H}_R^*$ . For all  $F \in \mathbf{F}$ , we put:

$$Z_F : \begin{cases} \mathcal{H}_R \longrightarrow K \\ G \longrightarrow s_F \delta_{F, G}. \end{cases}$$

As  $(Z_F)_{F \in \mathbf{F}(n)}$  is a basis of  $\mathcal{H}_R(n)^*$ ,  $(Z_F)_{F \in \mathbf{F}}$  is a basis of  $\mathcal{H}_R^*$ .

**Theorem 4.** *For any forest  $F, G \in \mathbf{F}$ , in  $\mathcal{H}_R^*$ :*

$$Z_F Z_G = \sum_{H \in \mathbf{F}} n'(F, G; H) Z_H.$$

For any  $t_1 \dots t_n \in \mathbf{F}$ :

$$\Delta(Z_{t_1 \dots t_n}) = \sum_{I \subseteq \{1, \dots, n\}} Z_{t_I} \otimes Z_{t_{\{1, \dots, n\} - I}},$$

where  $t_J = \prod_{j \in J} t_j$  for all  $J \subseteq \{1, \dots, n\}$ .

*Proof.* We put  $Z_F Z_G = \sum a_{F, G}^H Z_H$  in  $\mathcal{H}_R^*$ . For all  $F, G, H \in \mathbf{F}$ :

$$\begin{aligned} (Z_F Z_G)(H) &= a_{F, G}^H s_H \\ &= (Z_F \otimes Z_G) \circ \Delta(H) \\ &= (Z_F \otimes Z_G) \left( \sum_{A, B \in \mathbf{F}} n(A, B; H) A \otimes B \right) \\ &= n(F, G; H) s_F s_G. \end{aligned}$$



In particular,  $\begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array}$  and  $\begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array}$  are equal as rooted trees, but not as planar rooted trees.

### 2.2 The Hopf algebra of planar rooted trees

The Hopf algebra of planar rooted tree  $\mathcal{H}_{PR}$  was introduced simultaneously in Foissy [2002c] and Holtkamp [2003]. As an algebra,  $\mathcal{H}_{PR}$  is the free associative unitary algebra generated by  $\mathbf{T}_P$ . In other terms, a  $K$ -basis of  $\mathcal{H}_R$  is given by planar rooted forests, that is to say non necessarily connected graphs  $F$  such that each connected component of  $F$  is a planar rooted tree, and the roots of these rooted trees are totally ordered. The set of planar rooted forests will be denoted by  $\mathbf{F}_P$ . For all  $n \in \mathbb{N}$ , the set of rooted forests of weight  $n$  will be denoted by  $\mathbf{F}_P(n)$ . The product of  $\mathcal{H}_{PR}$  is given by the concatenation of planar rooted forests, and the unit is the empty forest, denoted by 1.

If  $t$  is a planar tree and  $c$  is an admissible cut of  $c$ , then the rooted tree  $R^c(t)$  is naturally a planar tree. Moreover, as  $c$  is admissible, the different rooted trees of the forest  $P^c(t)$  are planar and totally ordered from left to right, so  $P^c(t)$  is a planar forest. We then define a coproduct on  $\mathcal{H}_{PR}$  as the unique algebra morphism from  $\mathcal{H}_{PR}$  to  $\mathcal{H}_{PR} \otimes \mathcal{H}_{PR}$  such that, for all planar rooted tree  $t \in \mathbf{T}_P$ :

$$\Delta(t) = \sum_{c \in \text{Adm}_*(t)} P^c(t) \otimes R^c(t) = t \otimes 1 + 1 \otimes t + \sum_{c \in \text{Adm}(t)} P^c(t) \otimes R^c(t).$$

As  $\mathcal{H}_{PR}$  is the free algebra generated by  $\mathbf{T}_P$ , this makes sense.

*Examples 9.*

$$\begin{aligned} \Delta(\begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array}) &= \begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array} + \downarrow \otimes \downarrow + \dots \otimes \vee + \dots \otimes \downarrow + \downarrow \otimes \dots + \dots \otimes \downarrow, \\ \Delta(\begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array}) &= \begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow \\ \vee \\ \downarrow \end{array} + \downarrow \otimes \downarrow + \dots \otimes \vee + \dots \otimes \downarrow + \downarrow \otimes \dots + \dots \otimes \downarrow. \end{aligned}$$

**Theorem 5.** *With this coproduct,  $\mathcal{H}_{PR}$  is a bialgebra. The counit of  $\mathcal{H}_{PR}$  is given by:*

$$\varepsilon : \begin{cases} \mathcal{H}_{PR} \longrightarrow K \\ F \in \mathbf{F} \longrightarrow \delta_{1,F}. \end{cases}$$

The proof is the same as in the commutative case. In particular, we define an operator also denoted by  $B^+ : \mathcal{H}_{PR} \longrightarrow \mathcal{H}_{PR}$ , which associates to a planar forest  $t_1 \dots t_n$  the planar rooted tree obtained by grafting the different trees  $t_1, \dots, t_n$  on a common root, keeping the total order on their roots. For example,  $B^+(\downarrow \dots) = \begin{array}{c} \downarrow \\ \vee \\ \downarrow \dots \end{array}$ ,  $B^+(\dots \downarrow) = \begin{array}{c} \downarrow \\ \vee \\ \dots \downarrow \end{array}$  and  $B^+(\dots \downarrow) = \begin{array}{c} \downarrow \\ \vee \\ \dots \downarrow \end{array}$ . Similarly, it

is a 1-cocycle of  $\mathcal{H}_{PR}$ .

We put  $\mathcal{H}_{PR}(n) = Vect(\mathbf{F}_P(n))$ . This defines a connected gradation of  $\mathcal{H}_{PR}$ . For all  $n \in \mathbb{N}$ ,  $\dim_K(\mathcal{H}_{PR}(n))$  is the number of planar rooted forests of weight  $n$ , that is to say the  $n$ -th Catalan number:

**Proposition 2.** *For all  $n \in \mathbb{N}$ , we put  $R_n = \dim_K(\mathcal{H}_{PR}(n))$ . Then:*

$$\sum_{n=0}^{\infty} R_n h^n = \frac{1 - \sqrt{1 - 4h}}{2h}.$$

As a consequence,  $R_n = \frac{(2n)!}{(n+1)!n!}$  for all  $n \in \mathbb{N}$ .

The sequence  $(R_n)_{n \geq 0}$  is the sequence A000108 of Sloane.

*Example 10.*

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$r_n$	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	2674440

By lemma 2,  $\mathcal{H}_{PR}$  is a Hopf algebra. The antipode can also be described with cuts, although it is necessary to pay attention of the order of the trees in  $W^c(t)$ , as  $\mathcal{H}_{PR}$  is not commutative (Foissy [2002c]).

Similarly to the commutative case,  $\mathcal{H}_{PR}$  satisfies a universal property:

**Theorem 6 (Universal property of  $\mathcal{H}_{PR}$ ).** *Let  $A$  be an algebra and let  $L : A \rightarrow A$  be a linear application.*

1. *There exists a unique algebra morphism  $\phi : \mathcal{H}_{PR} \rightarrow A$ , such that  $\phi \circ B^+ = L \circ \phi$ .*
2. *If  $A$  is a Hopf algebra and  $L$  satisfies (1), then  $\phi$  is a Hopf algebra morphism.*

This property is more useful here than in the commutative case: we are going to use it to prove that  $\mathcal{H}_{PR}$  and its dual  $\mathcal{H}_{PR}^*$  are isomorphic.

### 2.3 Dual Hopf algebra and self-duality

For any  $F \in \mathbf{F}_P$ , we define the following element of the graded dual  $\mathcal{H}_{PR}^*$ :

$$Z_F : \begin{cases} \mathcal{H}_{PR} \rightarrow K \\ G \in \mathbf{F}_P \rightarrow \delta_{F,G} \end{cases}$$

Then  $(Z_F)_{F \in \mathbf{F}_P}$  is a basis of  $\mathcal{H}_{PR}^*$ . The coproduct of  $\mathcal{H}_{PR}^*$  is given by:

$$\Delta(Z_{t_1 \dots t_n}) = \sum_{i=0}^n Z_{t_1 \dots t_i} \otimes Z_{t_{i+1} \dots t_n}.$$

The product of  $Z_F$  and  $Z_G$  is given by planar graftings, similarly with the commutative case. The main difference is that there is several way to graft a planar tree on a vertex of a planar forest, and this implies the use of angles of a planar forest (Chapoton and Livernet [2001]).

*Example 11.*

$$Z \cdot Z = Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + Z \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}.$$

In order to prove the self-duality of  $\mathcal{H}_{PR}$ , we introduce the application  $\gamma$ :

$$\gamma : \begin{cases} \mathcal{H}_{PR} \longrightarrow \mathcal{H}_{PR} \\ t_1 \dots t_n \in \mathbf{F}_P \longrightarrow t_1 \dots t_{n-1} \delta_{t_n, \bullet} \end{cases}$$

$\gamma$  is clearly homogeneous of degree  $-1$ , so its transpose  $\gamma^* : \mathcal{H}_{PR}^* \longrightarrow \mathcal{H}_{PR}^*$  exists and is homogeneous of degree  $+1$ . It has the following properties:

- Lemma 4.** 1.  $\gamma^*$  is a 1-cocycle of  $\mathcal{H}_{PR}^*$ .  
 2.  $\mathcal{H}_{PR}^*$  is generated, as an algebra, by  $Im(\gamma^*)$ .

*Proof.* It is immediate that, for all planar forests  $F$  and  $G$ :

$$\gamma(FG) = F\gamma(G) + \varepsilon(G)\gamma(F).$$

So, by duality, identifying  $(\mathcal{H}_{PR} \otimes \mathcal{H}_{PR})^*$  and  $\mathcal{H}_{PR}^* \otimes \mathcal{H}_{PR}^*$ , if  $f \in \mathcal{H}_{PR}^*$ , for all planar forests  $F$  and  $G$ :

$$\begin{aligned} (\Delta \circ \gamma^*(f))(F \otimes G) &= (\gamma^*(f))(FG) \\ &= f \circ \gamma(FG) \\ &= f(F\gamma(G) + \varepsilon(G)\gamma(F)) \\ &= (\Delta(f))(F \otimes \gamma(G)) + (f \otimes 1)(\gamma(F) \otimes G) \\ &= ((Id \otimes \gamma^*) \circ \Delta(f) + \gamma^*(f) \otimes 1)(F \otimes G). \end{aligned}$$

This gives:

$$\Delta \circ \gamma^*(f) = (Id \otimes \gamma^*) \circ \Delta(f) + \gamma^*(f) \otimes 1,$$

so  $\gamma^*$  is a 1-cocycle of  $\mathcal{H}_{PR}^*$ .

Let us now prove that  $Im(\gamma^*)$  generates  $\mathcal{H}_{PR}^*$ . First, for all planar forests  $F, G = t_1 \dots t_n \in \mathbf{F}_P$ :

$$(\gamma^*(Z_F))(G) = Z_F(\delta_{t_n, \bullet} t_1 \dots t_{n-1}) = \delta_{F, t_1 \dots t_{n-1}} \delta_{t_n, \bullet} = \delta_{F, \bullet, G} = Z_{F \bullet}(G),$$

so  $\gamma^*(Z_F) = Z_{F\bullet}$  and  $Im(\gamma^*) = Vect(Z_{F\bullet} / F \in \mathbf{F}_P)$ . We denote by  $A$  the subalgebra of  $\mathcal{H}_{PR}^*$  generated by  $Im(\gamma^*)$ . Let  $G = t_1 \dots t_n \in \mathbf{F}_P$  and let us show that  $Z_G \in A$  by induction on  $p(F) = weight(t_n)$ . If  $p(F) = 1$ , then  $t_n = \bullet$  and  $Z_G \in A$ . If  $p(F) \geq 2$ , we put  $t_n = B^+(s_1 \dots s_m)$ . By the induction hypothesis,  $Z_{s_1 \dots s_m} \in A$ . So:

$$Z_{s_1 \dots s_m} Z_{t_1 \dots t_{n-1}\bullet} = Z_{t_1 \dots t_n} + \text{linear span of } Z_G \text{ with } p(G) < p(F) \in A.$$

By the induction hypothesis,  $Z_F \in A$ . So  $A = \mathcal{H}_{PR}^*$ .  $\square$

As  $\gamma^*$  is a 1-cocycle of  $\mathcal{H}_{PR}^*$ , by the universal property of  $\mathcal{H}_{PR}$  there exists a unique Hopf algebra morphism  $\phi : \mathcal{H}_{PR} \longrightarrow \mathcal{H}_{PR}^*$ , such that  $\phi \circ B^+ = \gamma^* \circ \phi$ .

**Theorem 7.**  $\phi$  is an isomorphism, homogeneous of degree 0.

*Proof.* Let us first prove that  $\phi$  is homogeneous: we show that for any forest  $F \in \mathbf{F}_P(m)$ ,  $\phi(F)$  is homogeneous of degree  $m$  by induction on  $m$ . If  $m = 0$ , then  $F = 1$  and the result is obvious. If  $m \geq 2$ , two cases can occur. First,  $F = t_1 \dots t_n$ , with  $n \geq 2$ . Then the induction hypothesis can be applied to the  $t_i$ 's, so  $\phi(F) = \phi(t_1) \dots \phi(t_n)$  is homogeneous of degree  $weight(t_1) + \dots + weight(t_n) = weight(F)$ . Secondly, if  $F = B^+(G)$ , then the induction hypothesis can be applied to  $G$ . So  $\phi(F) = \gamma^* \circ \phi(G)$  is homogeneous of degree  $weight(G) + 1 = weight(F)$ , as  $\gamma^*$  is homogeneous of degree 1.

Let us show that  $\phi$  is epic. We consider the following assertions:

$P_n : Im(\phi)$  contains  $\mathcal{H}_{PR}^*(k)$  for all  $k \leq n$ .

$Q_n : Im(\phi)$  contains  $\gamma^*(\mathcal{H}_{PR}^*(k))$  for all  $k \leq n$ .

Let us prove that  $P_n \implies Q_n$ . Let  $x \in \mathcal{H}_{PR}^*(k)$ ,  $k \leq n$ . By  $P_n$ ,  $x = \phi(y)$  for a  $y \in \mathcal{H}_{PR}$ . Then  $\phi \circ B^+(x) = \gamma^* \circ \phi(y) = \gamma^*(x)$ , so  $Q_n$  is true. Let us show that  $Q_n \implies P_{n+1}$ . Let  $x \in \mathcal{H}_{PR}^*(k)$ ,  $k \leq n + 1$ . As  $Im(\gamma^*)$  generates  $\mathcal{H}_{PR}^*$ ,  $x$  can be written under the form:

$$x = \sum_k \gamma^*(x_{k,1}) \dots \gamma^*(x_{k,n_k}).$$

By homogeneity, as  $\gamma^*$  is homogeneous of degree 1, we can suppose that all the  $x'_{i,j}$ 's are homogeneous of degree  $\leq n$ . By  $Q_n$ , the  $\gamma^*(x_{i,j})$ 's belong to  $Im(\phi)$ . As  $\phi$  is an algebra morphism,  $Im(\phi)$  is a subalgebra of  $\mathcal{H}_{PR}^*$ , so  $x \in Im(\phi)$ .

As a conclusion,  $P_n \implies Q_n \implies P_{n+1}$ . As  $P_0$  is clearly true,  $P_n$  is true for all  $n$ , so  $\phi$  is epic. As it is also homogeneous of degree 0 and the homogeneous components of degree  $n$  of  $\mathcal{H}_{PR}$  and  $\mathcal{H}_{PR}^*$  have the same finite dimension,  $\phi$  is also monic.  $\square$

There are two alternative ways to see this isomorphism. The first one is in term of Hopf pairing. We put, for all  $x, y \in \mathcal{H}_{PR}$ ,  $\langle x, y \rangle = \phi(x)(y)$ . As  $\phi$  is a Hopf algebra morphism, this pairing satisfies the following properties:





## 2.4 A link with the commutative case

We consider:

$$\varpi : \begin{cases} \mathcal{H}_{PR} \longrightarrow \mathcal{H}_R \\ F \in \mathbf{F}_P \longrightarrow \text{the underlying rooted forest of } F. \end{cases}$$

This is clearly an epimorphism of Hopf algebras, homogeneous of degree 0. Dually, we obtain a monomorphism of Hopf algebras:

$$\varpi^* : \begin{cases} \mathcal{H}_R^* \longrightarrow \mathcal{H}_{PR}^* \\ Z_F, F \in \mathbf{F} \longrightarrow \frac{1}{s_F} \sum Z_{\tilde{F}}, \end{cases}$$

where the sum is taken over the planar rooted forests  $\tilde{F}$  with underlying rooted forest  $F$ . Using the isomorphism  $\phi$ , we obtain that the subspace of  $\mathcal{H}_{PR}$  with basis  $(\sum e_{\tilde{F}})_{F \in \mathbf{F}}$ , where the sum is taken in the same way, is a subalgebra of  $\mathcal{H}_{PR}$  isomorphic to  $\mathcal{H}_R^*$ .

## 3 Non associative algebraic structures and applications

### 3.1 PreLie structures on $\mathcal{H}_R^*$

By the Milnor-Moore theorem (Milnor and Moore [1965]),  $\mathcal{H}_R^*$ , being a graded, connected, cocommutative Hopf algebra, is isomorphic to the enveloping algebra of its primitive elements. Let us now consider the Lie algebra of primitive elements of  $\mathcal{H}_R^*$ . By theorem 4, a basis of  $Prim(\mathcal{H}_R^*)$  is given by  $(Z_t)_{t \in \mathbf{T}}$ . Moreover, if  $t_1, t_2 \in \mathbf{T}$ , still with theorem 4:

$$Z_{t_1} Z_{t_2} = \sum_{F \in \mathbf{F}} n'(t_1, t_2; F) Z_F.$$

Note that, if  $n'(t_1, t_2; F) \neq 0$ , then  $F = t_1 t_2$  or  $F$  is a tree. As a consequence:

$$\begin{aligned} Z_{t_1} Z_{t_2} &= Z_{t_1 t_2} + \sum_{t \in \mathbf{T}} n'(t_1, t_2; t) Z_t, \\ [Z_{t_1}, Z_{t_2}] &= \sum_{t \in \mathbf{T}} n'(t_1, t_2; t) Z_t - \sum_{t \in \mathbf{T}} n'(t_2, t_1; t) Z_t. \end{aligned}$$

We then define a product  $\circ$  on  $Prim(\mathcal{H}_R^*)$  by:

$$Z_{t_1} \circ Z_{t_2} = \sum_{t \in \mathbf{T}} n'(t_1, t_2; t) Z_t.$$

This product is not associative, but satisfies the following identity: for all  $x, y, z \in Prim(\mathcal{H}_R^*)$ ,

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z),$$

that is to say  $(Prim(\mathcal{H}_R^*), \circ)$  is a (left) preLie algebra, or equivalently a left Vinberg algebra, or a left-symmetric algebra (Chapoton [2001], Chapoton and Livernet [2001], van der Laan and Moerdijk [2006]). It is proved in Chapoton and Livernet [2001] that  $(Prim(\mathcal{H}_R^*), \circ)$  is freely generated by  $Z_\bullet$  as a preLie algebra. This result is proved using a tree-description of the operad of preLie algebras. Moreover, this product  $\circ$  can be extended to  $S(Prim(\mathcal{H}_R^*))$ , making it isomorphic to  $\mathcal{H}_R^*$  (Oudom and Guin [2005]).

### 3.2 Application: two Hopf subalgebras of $\mathcal{H}_R$

Let  $(\mathfrak{g}, \circ)$  be a graded preLie algebra, generated by a single element  $x$ , homogeneous of degree 1. As  $Prim(\mathcal{H}_R^*)$  is freely generated by  $Z_\bullet$ , there exists a unique morphism of preLie algebras from  $Prim(\mathcal{H}_R^*)$  to  $\mathfrak{g}$ , sending  $Z_\bullet$  to  $x$ . As  $x$  generates  $\mathfrak{g}$ , this morphism is epic. As  $x$  is homogeneous of degree 1, this morphism is homogeneous of degree 0. It can be extended in a Hopf algebra morphism  $\phi : \mathcal{U}(Prim(\mathcal{H}_R^*)) \approx \mathcal{H}_R^* \rightarrow \mathcal{U}(\mathfrak{g})$ , epic and homogeneous of degree 0. Dually, its transposition is a monomorphism of Hopf algebras  $\phi^* : \mathcal{U}(\mathfrak{g})^* \rightarrow \mathcal{H}_R$ . We obtain in this way Hopf subalgebras of  $\mathcal{H}_R$ , as the two following examples.

For the first example, we take  $\mathfrak{g}_{ladders} = Vect(Z_i / i \in \mathbb{N}^*)$ , with the product given by  $Z_i \circ Z_j = Z_{i+j}$ . This product is associative, so is preLie. It is commutative, so the induced Lie bracket on  $\mathfrak{g}_{ladders}$  is trivial. Moreover,  $\mathfrak{g}_{ladders}$  is graded by putting  $Z_i$  homogeneous of degree  $i$ , and is generated by  $Z_1$ . So there is an epimorphism  $\phi_{ladders}$  of preLie algebras for  $Prim(\mathcal{H}_R^*)$  to  $\mathfrak{g}_{ladders}$ , sending  $Z_\bullet$  to  $Z_1$ .

*Notations 2.* For all  $n \in \mathbb{N}$ , we put  $l_n = (B^+)^n(1)$  (ladder of weight  $n$ ). For example,  $l_1 = \bullet$ ,  $l_2 = \vdots$ ,  $l_3 = \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix}$ ,  $l_4 = \begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ .

**Lemma 5.** *The preLie algebra morphism  $\phi_{ladders}$  is given by:*

$$\phi_{ladders} : \begin{cases} Prim(\mathcal{H}_R^*) \longrightarrow \mathfrak{g}_{ladders} \\ Z_t \longrightarrow 0 \text{ if } t \text{ is not a ladder,} \\ Z_{l_n} \longrightarrow Z_n. \end{cases}$$

*Proof.* It is enough to prove that the thus defined linear application is indeed a preLie algebra morphism. Let  $t$  and  $t'$  be two elements of  $\mathbf{T}$ . If  $t$  or  $t'$  is not a ladder, then there is no grafting of  $t$  on  $t'$  giving a ladder, so  $\phi_{ladders}(t \circ t') = 0 = \phi_{ladders}(t) \circ \phi_{ladders}(t')$ . If  $t = l_m$  and  $t' = l_n$ , then there is a unique grafting of  $t$  on  $t'$  giving a ladder, which is  $l_{m+n}$ . So  $\phi_{ladders}(t \circ t') = Z_{m+n} = Z_m \circ Z_n = \phi_{ladders}(t) \circ \phi_{ladders}(t')$ .  $\square$

Dually,  $\phi_{ladders}^*$  is the algebra morphism sending  $Z_n^*$  to  $l_n$  for all  $n \geq 1$ . So the image of  $\phi_{ladders}^*$  is the subalgebra of  $\mathcal{H}_R$  generated by ladders, which is indeed a Hopf subalgebra: for all  $n \geq 1$ ,

$$\Delta(l_n) = \sum_{i=0}^n l_i \otimes l_{n-i}.$$

This is a commutative, cocommutative Hopf algebra, isomorphic to the Hopf algebra of symmetric functions (Duchamp et al. [2002], Stanley [1999]).

For the second example, we take  $\mathfrak{g}_{FdB} = Vect(Z_i / i \in \mathbb{N}^*)$ , with the product given by  $Z_i \circ Z_j = jZ_{i+j}$ . This product is preLie: for all  $i, j, k \in \mathbb{N}^*$ ,

$$\begin{aligned} (Z_i \circ Z_j) \circ Z_k - Z_i \circ (Z_j \circ Z_k) &= jkZ_{i+j+k} - (j+k)kZ_{i+j+k} \\ &= -k^2Z_{i+j+k} \\ &= (Z_j \circ Z_i) \circ Z_k - Z_j \circ (Z_i \circ Z_k). \end{aligned}$$

Moreover,  $\mathfrak{g}_{FdB}$  is graded by putting  $Z_i$  homogeneous of degree  $i$ , and is clearly generated by  $Z_1$ . So there is an epimorphism  $\phi_{FdB}$  of preLie algebras for  $Prim(\mathcal{H}_R^*)$  to  $\mathfrak{g}_{FdB}$ , sending  $Z_\bullet$  to  $Z_1$ .

**Lemma 6.** *The preLie algebra morphism  $\phi_{FdB}$  is given by:*

$$\phi_{FdB} : \begin{cases} Prim(\mathcal{H}_R^*) \longrightarrow \mathfrak{g}_{FdB} \\ Z_t \longrightarrow Z_{weight(t)}. \end{cases}$$

*Proof.* It is enough to prove that the thus defined linear application is indeed a preLie algebra morphism. Let  $t$  and  $t'$  be two elements of  $\mathbf{T}$ , of respective weights  $n$  and  $n'$ . There are exactly  $n'$  graftings of  $t$  on  $t'$ , so  $\phi_{FdB}(t \circ t') = n'Z_{n+n'} = Z_n \circ Z_{n'} = \phi_{FdB}(t) \circ \phi_{FdB}(t')$ .  $\square$

Dually,  $\phi_{FdB}^*$  is the algebra morphism sending  $Z_n^*$  to  $\delta_n$ , defined by:

$$\delta_n = \sum_{weight(t)=n} \frac{1}{s_t} t.$$

So the image of  $\phi_{FdB}^*$  is the subalgebra of  $\mathcal{H}_R$  generated by the  $\delta_n$ 's, which is consequently a Hopf subalgebra. This is one of the subalgebra of Foissy [2008], coming from a Dyson-Schwinger equation, and is isomorphic to the Faà di Bruno Hopf algebra (Figueroa et al. [2005]). Note that another imbedding of the Faà di Bruno, known as the Connes-Moscovici subalgebra, is given in Connes and Kreimer [1998], with the notion of growth, or equivalently of heap-orderings of rooted trees. For example, its first generators are:

$$\begin{aligned} \delta'_1 &= \bullet, \\ \delta'_2 &= \mathbf{!}, \\ \delta'_3 &= \mathbf{V} + \mathbf{!}, \\ \delta'_4 &= \mathbf{V} + 3\mathbf{!} + \mathbf{!} + \mathbf{!}. \end{aligned}$$

### 3.3 Dendriform structures on $\mathcal{H}_{PR}$

The notion of dendriform algebra is introduced in Loday [2001]. Namely, this is an associative algebra  $(A, *)$ , such that  $*$  can be written as  $* = \prec + \succ$ , with the following compatibilities: for all  $x, y \in A$ ,

$$\begin{cases} x \prec (y \prec z) = (x * y) \succ z, \\ x \succ (y \prec z) = (x \succ y) \prec z, \\ x \succ (y * z) = (x \succ y) \succ z. \end{cases}$$

In other terms,  $(A, \prec, \succ)$  is a bimodule over  $(A, *)$ . Note that dendriform algebras are not unitary objects. The free dendriform algebra on one generator is described in Loday and Ronco [1998] in terms of planar binary trees, obtaining a Hopf algebra on these objects known as the Loday-Ronco Hopf algebra (see also Aguiar and Sottile [2006]). It is shown in Foissy [2002b] that this Hopf algebra is isomorphic to  $\mathcal{H}_{PR}$ , and as a corollary, the augmentation ideal  $\mathcal{H}_{PR}^+$  of  $\mathcal{H}_{PR}$  inherits a structure of dendriform algebra, given in the dual basis  $(e_F)_{F \in \mathbf{F}_P}$  in terms of graftings. As the product  $e_F e_G$  is given by the graftings of  $F$  over  $G$  (by similarity with  $\mathcal{H}_{PR}^*$ ), the left product is given by graftings of  $F$  over  $G$  such that the last tree of the grafting is the last tree of  $F$ . In particular, for all  $t \in \mathbf{T}$ ,  $F \in \mathbf{F}$ ,  $e_t \prec e_F = e_{Ft}$ . For this dendriform structure,  $\mathcal{H}_{PR}^+$  is freely generated by  $\dots$

*Example 14.*

$$\begin{aligned} e_{\dots} e_{\downarrow} &= e_{\dots \downarrow} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \vee} \\ &\quad + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \downarrow}, \\ e_{\dots} \prec e_{\downarrow} &= e_{\dots \downarrow} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \vee} + e_{\downarrow \downarrow}, \\ e_{\dots} \succ e_{\downarrow} &= e_{\dots \downarrow} + e_{\downarrow \vee} + e_{\downarrow \downarrow} + e_{\downarrow \vee} \\ &\quad + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \vee} + e_{\downarrow \vee}. \end{aligned}$$

Dually, it is also possible to cut the coproduct of  $\mathcal{H}_{PR}$  into two parts, with good compatibilities with the left and right products. The obtained result is called a bidendriform bialgebra (Foissy [2007]). This formalism, together with a rigidity theorem, allows to prove for example that the Malvenuto-Reutenauer Hopf algebra, also known as the Hopf algebra of free quasi-symmetric functions (Duchamp et al. [2002], Malvenuto and Reutenauer [1995]) is isomorphic to a decorated version of  $\mathcal{H}_{PR}$ .

Using the dendriform Milnor-Moore theorem of Loday and Ronco [1998], any connected dendriform Hopf algebra can be seen as the dendriform enveloping algebra of a brace algebra. As in the commutative case, replacing

preLie algebras by brace algebras, it is possible to construct some Hopf subalgebras of  $\mathcal{H}_{PR}$ . In particular, the subalgebra generated by the ladders is a non commutative, cocommutative Hopf subalgebra isomorphic to the Hopf algebra of non commutative symmetric functions (Duchamp et al. [2002]), or to a bitensorial Hopf algebra (Manchon [1997]). It is also possible to give non commutative versions of the Faà di Bruno subalgebras, for example the subalgebra generated in degree  $n$  by the sum of all planar trees of weight  $n$  (Foissy [2002b, 2008]).

## Conclusion

By way of conclusion, we would like to mention that the Hopf algebras  $\mathcal{H}_R$  and  $\mathcal{H}_{PR}$  has appeared in several areas. First, following Connes and Kreimer [1998], eventually working with Hopf algebras of Feynman graphs, the applications to the Renormalization is explored in Bergbauer and Kreimer [2005, 2006], Broadhurst and Kreimer [2000b,a], Chryssomalakos et al. [2002], Connes and Kreimer [2000, 2001a,b], Ebrahimi-Fard et al. [2004, 2005], Figueroa and Gracia-Bondia [2001, 2004], Krajewski and Wulkenhaar [1999], Kreimer and Delbourgo [1999], Kreimer [1999a,b, 2002].

Applications of the Birkhoff decomposition on characters group for connected Hopf algebra are given in Brouder and Schmitt [2007], Cartier [2007], Girelli et al. [2004], Manchon [2004], Turaev [2005].

Non-commutative versions of Hopf algebras of Renormalization, based on planar binary trees, are described in Brouder and Frabetti [2003], Byun [2005], Erjavec [2006].

The Hopf algebra  $\mathcal{H}_R$  is also related to the Butcher group of Runge-Kutta methods, as shown in Brouder [2004], and to the process of arborification-coarborification in Ecalle's mould calculus, as explained in Menous [2007].

From an algebraic point of view,  $\mathcal{H}_R$  and  $\mathcal{H}_{PR}$  and their extra structures are related to operads and free objects in Chapoton [2001], Chapoton and Livernet [2001], Moerdijk [2001], Murua [2006], Oudom and Guin [2005], van der Laan and Moerdijk [2006], and to other combinatorial Hopf algebras in Aguiar and Sottile [2005], Hoffman [2003], Holtkamp [2003], Panaite [2000].

Several algebraic results (self-duality of  $\mathcal{H}_{PR}$ , comodules, Hopf subalgebras, etc) are given in Foissy [2002a,c,b, 2008], Zhao [2004] and a quantization of a decorated version of  $\mathcal{H}_{PR}$  is described in Foissy [2003].

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