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MILNOR-MOORE THEOREMS FOR BIALGEBRAS IN CHARACTERISTIC ZERO.

JOEY BEAUVAIS-FEISTHAUER, YATIN PATEL, ANDREW SALCH

ABSTRACT. Over fields of characteristic zero, we construct equivalences between certain categories of bialgebras which are generated by grouplikes and generalized primitives, and certain categories of structured Lie algebras. The relevant families of bialgebras include many which are not connected, and which fail to admit antipodes.

CONTENTS

1. INTRODUCTION

1.1. **The Milnor-Moore theorem, and some generalizations.** Let k be a field of characteristic zero. The characteristic zero case of the classical theorem of Milnor and Moore [23] tells us that the category of Lie k-algebras is equivalent to the category of primitively-generated Hopf k-algebras. The equivalence of categories is given by the two functors

$$P : \mathrm{PrimGen\,Hopf\,Alg}(k) \to \mathrm{Lie}(k)$$
$$U : \mathrm{Lie}(k) \to \mathrm{PrimGen\,Hopf\,Alg}(k),$$

where P sends a Hopf algebra to its Lie algebra of primitives, and where U sends a Lie algebra to its universal enveloping algebra. While many generalizations of this theorem can be found in the literature[1], we have not been able to find any generalizations to bialgebras which are not necessarily connected, and which do not necessarily admit antipodes, i.e., bialgebras which are not necessarily Hopf algebras. The purpose of this paper is to formulate and prove several such generalizations. We are particularly focused on the bialgebras with non-invertible grouplike elements. Such bialgebras necessarily cannot admit antipodes. While we think these generalizations of the Milnor-Moore theorem are of some interest for purely algebraic reasons, we also have some motivation from examples in algebraic topology; see the end of this introduction for the relevance to topology.

It is clear that a sufficiently naïve attempt to generalize the Milnor-Moore theorem to bialgebras cannot possibly succeed. If we are to generalize the Milnor-Moore theorem to bialgebras, we must decide what category appears on the "Lie side" of the correspondence, and what category appears on the "bialgebra side" of the correspondence. Primitively-generated bialgebras, however, are Hopf algebras; see section 5, below, for a proof (which is certainly not new). So the bialgebra side of the equivalence cannot simply be primitively-generated bialgebras.

We propose that the bialgebra side ought to be the *generalized-primitively-generated* bialgebras. Recall that a primitive in a bialgebra A is an element $a \in A$ such that $\Delta(a) = a \otimes 1 + 1 \otimes a$. Meanwhile, a grouplike in A is an element $a \in A$ such that $\Delta(a) = a \otimes a$ and $\varepsilon(a) = 1$. If Q is a grouplike element of A, we define a *Q-primitive*[2] to be an element $a \in A$ such that $Q \otimes a + a \otimes Q$. A *generalized primitive* is a linear combination of Q-primitives for various grouplikes $Q \in A$. Finally, a bialgebra is *generalized-primitively-generated* if it is generated, as an algebra, by grouplikes and generalized primitives[3].

Now recall that, for a bialgebra A, the set of grouplikes $\Gamma(A)$ of A forms a monoid under multiplication. When we turn to the Lie side of our equivalence of categories, we will find that the relevant type of extra structure on the Lie algebras will depend on the structure of the monoid $\Gamma(A)$. So it is convenient, on the bialgebra side of our equivalence of categories, to consider not only the generalized-primitively-generated bialgebras. Instead, we fix a commutative monoid G, and we want to work with bialgebras A which are generalized-primitively-generated *and equipped with a choice*

[1]See for example [4], [5], [10], [13], [17], [29], and [30]. Some of these references are written in terms of bialgebras or generalizations of bialgebras, but as far as we have been able to determine, these and all other currently-available generalizations of the Milnor-Moore theorem make an assumption of one kind or another which amounts, in the classical setting of *bialgebras* (rather than an abstract generalization of bialgebras in some categorical setting), to connectedness, hence admitting an antipode. Connectedness assumptions are quite reasonable if one is already willing to assume the existence of an antipode, since in the presence of an antipode, the connected part and the grouplikes can be treated separately. In the bialgebra setting, no such decomposition is available, so the grouplikes and primitives have to be treated in a "mixed" way, as the reader can see from the methods developed in this paper.

[2]This is a special case of a "skew-primitive," an element a such that $\Delta(a) = Q \otimes a + a \otimes Q'$ for grouplikes Q, Q' of A. Skew-primitives are well-studied, appearing already in Bourbaki's volume on Lie theory [7].

[3]Generalized-primitively-generated bialgebras are automatically co-commutative; see Proposition 2.12. So to be generalized-primitively-generated is equivalent to being co-commutative and generated by grouplikes and skew-primitives. The condition of being generated by grouplikes and skew-primitives is a natural one, due to substantial interest in the Andruskiewitsch-Schneider conjecture [1], i.e., the conjecture that every finite-dimensional Hopf algebra over an algebraically closed field of characteristic zero is generated by grouplikes and skew-primitives.

of isomorphism $\Gamma(A) \cong G$. We call bialgebras with such a choice of isomorphism *G-rigid bialgebras*; see Definition 4.1 for the full definition[4].

Now we turn to the Lie side of our equivalence of categories. Fixing a commutative monoid G, we find that the set of generalized primitives of a G-rigid k-bialgebra A is not only a Lie algebra, but in fact a *G-suspensive Lie k-algebra*. A G-suspensive Lie k-algebra, defined in Definition 2.7, is a G-graded Lie k-algebra which is furthermore equipped with an action of G which respects the G-grading and such that the Lie bracket is kG-bilinear. Section 3 has some general remarks and discussion on the subject of this kind of "suspensive algebra," i.e., familiar algebraic gadgets like Lie algebras but which are equipped with both a G-grading and a G-action. The idea of "suspensive algebra" is elementary, but we do not know of anywhere where it has been examined before.

In Definition-Proposition 2.9, we define the *universal G-rigid enveloping algebra* WL of a G-suspensive Lie algebra L. In Proposition 4.2, we find that

$$W : \mathrm{Susp}_G \,\mathrm{Lie}(k) \to \mathrm{Cocomm}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k)$$

is left adjoint to the functor

$$GP_* : \mathrm{Cocomm}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k) \to \mathrm{Susp}_G\,\mathrm{Lie}(k)$$

that sends a cocommutative G-rigid k-bialgebra to its G-suspensive Lie k-algebra of generalized primitives. Our first generalization of the characteristic zero Milnor-Moore theorem is Theorem 6.9, which states that W and GP restrict to an equivalence of categories between

- *torsion-free* G-suspensive Lie k-algebras, and
- *torsion-free* generalized-primitively-generated G-rigid k-bialgebras.

A G-suspensive Lie k-algebra L is *torsion-free* if, for each nonzero homogeneous element $x \in L$ of degree $Q \in G$, we have $Q \cdot x \neq 0$. Similarly, a G-rigid bialgebra A is *torsion-free* if its G-suspensive Lie algebra of generalized primitives is torsion-free. This curious kind of "torsion" is only defined because G-suspensive Lie algebras are equipped with *both* a G-grading and a G-action. To be clear, when G is not a group, not every torsion-free G-suspensive Lie k-algebra is free (or projective) as a kG-module, so G-suspensive Lie algebras are not a special case of Lie R-algebras projective over R, as studied already in [23].

If the commutative monoid G *is* a group, then the whole theory reduces to the well-known classical one: every G-suspensive Lie algebra is torsion-free, and furthermore Theorem 6.9 is a case of the classical Milnor-Moore theorem. See Corollary 6.10 for discussion. We emphasize that the results in this paper, and in particular Theorem 6.9, are only of consequence in the case when the monoid G has noninvertible elements, i.e., the case of bialgebras with some noninvertible grouplikes.

Theorem 6.9 does require the torsion-freeness hypothesis, as we explain with an explicit example in Example 6.11. In section 7 we explain what would be necessary to formulate a *completely* general Milnor-Moore theorem for bialgebras, one that drops the torsion-freeness hypothesis. We explain that, in order for the "bialgebra

[4]To be clear, the definition of a G-rigid bialgebra includes the condition that the grouplikes $\Gamma(A)$ of A are contained in the center of A. Bialgebras with noncentral grouplikes do not lie within the scope of the Milnor-Moore-type theorems proven in this paper.

side" of the equivalence to be *all* generalized-primitively-generated G-rigid bialgebras, the "Lie side" would have to be G-suspensive Lie algebras equipped with a truly ponderous structure, an n-ary product for each $n = 2, 3, \dots$, satisfying some inconvenient associativity conditions as well as conditions enforcing compatibility with the Lie bracket and the suspensive structure. This structure is truly burdensome to work with, and we regard such a wide generalization of the Milnor-Moore theorem as unfruitful: one might as well just work with the bialgebras, if the Lie side of the equivalence is so complicated.

Nevertheless, there are other classes of bialgebras and suspensive Lie algebras, besides the torsion-free ones, for which a Milnor-Moore-type equivalence holds. In Theorem 8.9 we prove a Milnor-Moore-type theorem for *torsion* G-suspensive Lie algebras, i.e., G-suspensive Lie algebras L in which $Qx = 0$ for all homogeneous $x \in L$ in degree $Q \in G$. Such suspensive Lie algebras are at the opposite extreme from those considered in Theorem 6.9, which were torsion-*free*. Theorem 8.9 establishes that, when the commutative monoid G is linear[5], the category of torsion G-suspensive Lie k-algebras is equivalent to the category of *left-sided* generalized-primitively-generated G-rigid k-bialgebras. A G-rigid k-bialgebra A is *left-sided* if, for all $Q, Q' \in G$ such that Q divides Q', for all $x \in GP_Q(A)$ and all $y \in GP_{Q'}(A)$ we have $xy = 0$. Left-sidedness is a very strong condition, but satisfied by the associated graded bialgebra of a natural filtration on the Dyer-Lashof algebra, as explained in Proposition 8.1.

Throughout, we work over a field of characteristic zero. Presumably it is possible to generalize the results in this paper to fields of positive characteristic, by keeping track of a restriction map on the "Lie side" of each equivalence. We have found ourselves already with enough to say in characteristic zero that we chose not to pursue the positive-characteristic case in this paper.

1.2. **Topological examples and motivations.** There are two topological sources of non-Hopf bialgebras which motivated the authors to consider Milnor-Moore-like theorems for bialgebras:

(1) For each prime number p, the p-primary *Dyer-Lashof algebra*, written R, is an $\mathbb{N}$-graded co-commutative $\mathbb{F}_p$-bialgebra of operations on the mod p homology of infinite loop spaces. (There is a different Dyer-Lashof algebra R for each prime p, but the choice of prime p is suppressed from the notation.) The degree zero subring of R is isomorphic to $\mathbb{F}_p[Q^0]$ with Q^0 a noninvertible grouplike element, so the bialgebra R clearly cannot admit an antipode. See the appendix to this paper for a review of basic properties of R, including generators and relations.

    Basterra [6] and Miller [22] have constructed spectral sequences whose input involves Tor and Ext groups over the Dyer-Lashof algebra. Basterra's spectral sequences, in particular, are one of very few available tools for the calculation of topological André-Quillen cohomology groups of commutative ring spectra. While one can use Priddy's Koszul duality from [25] to obtain a complete description of $\mathrm{Ext}_R^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ (at $p = 2$, in [22], and all primes p in [15]), nevertheless one may hope to arrive a new (hopefully useful!)

[5] Linearity of G is defined in Definition 8.6. The motivating example of a linear commutative monoid G is the free commutative monoid $\mathbb{N}$, since this case appears prominently in motivating topological examples, like the Dyer-Lashof algebra.

perspective on the homological algebra of R-modules by understanding R in terms of Lie algebras.

Recall that, for each prime p, the Steenrod algebra A is a co-commutative Hopf $\mathbb{F}_p$-algebra. It is not the case that A is primitively-generated. In May's thesis [20], May filters A so that its associated graded Hopf algebra E_0A *is* primitively-generated. Consequently May gets a spectral sequence whose input is Ext over E_0A, and whose output is Ext over A, and since E_0A is primitively-generated, May is able to use techniques from Lie algebra cohomology[6] to understand and calculate Ext_{E_0A}.

Similarly, while the Dyer-Lashof algebra R is not *generalized*-primitively-generated, if we filter it by powers of the ideal consisting of all elements in positive degrees, its associated graded bialgebra *is* generalized-primitively-generated[7]. In fact E_0R is, at $p > 2$, a left-sided $\mathbb{N}$-rigid $\mathbb{F}_p$-bialgebra, so our Milnor-Moore-type theorem for left-sided rigid bialgebras, Theorem 8.9, *nearly* applies to E_0R. The only trouble is that Theorem 8.9 is for fields of characteristic zero! If the Lie-algebra-theoretic methods in this paper can indeed shed any light on homological questions about the Dyer-Lashof algebra, it will have to wait until positive-characteristic versions of our theorems are proven.

(2) We now sketch one more topological application of the ideas in this paper. Given a path-connected homotopy-associative unital H-space X, a classical result of Cartan and Serre [9] identifies the primitives in the rational homology $H_*(X;\mathbb{Q})$ of X as the image of the rational Hurewicz map $\pi_*(X)\otimes_{\mathbb{Z}}\mathbb{Q} \to H_*(X;\mathbb{Q})$. The Lie bracket of primitives in $H_*(X;\mathbb{Q})$ then agrees with the Samelson product on homotopy groups. A classical application of the characteristic zero Milnor-Moore theorem then proves:

Theorem 1.1. (Milnor-Moore; see appendix of [23]**.)** *If X is a path-connected homotopy-associative unital H-space, then the rational homology of X is isomorphic to the universal enveloping algebra of the rational homotopy $\pi_*(X)\otimes_{\mathbb{Z}}\mathbb{Q}$, regarded as a Lie algebra via the Samelson product.*

It is natural to ask how to generalize Theorem 1.1 to handle a homotopy-associative H-space X which is *not path-connected.* Here are some of the obstacles to formulating such a generalization:

Algebraic obstacle: If X is not path-connected, then $H_0(X;\mathbb{Q})$ will typically have nontrivial grouplikes. Then $H_*(X;\mathbb{Q})$ will not be primitively-generated, so the classical characteristic zero Milnor-Moore theorem cannot offer a complete description of $H_*(X;\mathbb{Q})$ in terms of some kind of Lie-algebraic data. We might deal with the grouplikes in $H_0(X;\mathbb{Q})$ by using the Cartier-Gabriel-Kostant-Milnor-Moore theorem which identifies every cocommutative Hopf algebra over an algebraically closed field of characteristic zero as the twisted tensor product

[6]The Lie-algebra-theoretic content of May's thesis is glossed over in some treatments, such as [28], where the focus is on using May's spectral sequence for explicit calculations of relevance for topology. See [18] and [21] for published accounts of the Lie-algebra-theoretic techniques and results from May's thesis.

[7]Note that, unlike the Steenrod algebra, the augmentation ideal of R is not the ideal of all elements in positive degrees. For example, $1 - Q^0$ is in the augmentation ideal, but also in degree zero.

of a group algebra and the universal enveloping algebra of some Lie algebra. However, this requires the Hopf algebra to indeed be a *Hopf* algebra, not just a bialgebra: the antipode plays an essential role in the proof of the Cartier-Gabriel-Kostant-Milnor-Moore theorem.

Our point is that, when X is a non-connected homotopy-associative unital H-space but not an H-group, we really need some analogue of the characteristic zero Milnor-Moore theorem which applies to bialgebras without antipode. Such a theorem is the main result of this paper.

Topological obstacle: In the proof of the Cartan-Serre theorem on primitives in $H_*(X;\mathbb{Q})$, an essential role is played by the fact that the rational k-invariants of a path-connected homotopy-associative unital H-space X are all trivial, so the rational homotopy type of X splits as a product of Eilenberg-Mac Lane spaces. See section 9.1 of [19] for a nice modern exposition. If we lift the hypothesis that X is path-connected, then we are no longer guaranteed to have such a splitting: for example, consider the case where X is the free topological monoid on any space with a rationally nontrivial k-invariant.

Our point is that there is a topological obstacle to identifying $H_*(X;\mathbb{Q})$ in terms of the Lie algebra $\pi_*(X)\otimes_{\mathbb{Z}}\mathbb{Q}$: one ought to formulate and prove some appropriate analogue of the Cartan-Serre theorem on primitives in $H_*(X;\mathbb{Q})$, in the case of X not path connected. Dealing with that obstacle goes beyond the scope of this paper, however.

1.3. **Acknowledgments.** This paper has its origins in conversations that A. Salch had with Sean Tilson several years ago, motivated by the desire to better understand the input for Basterra's spectral sequences from [6]. Salch thanks Tilson for those conversations and for his hospitality during a visit to Wuppertal.

2. Q-primitives.

2.1. **Basic definitions.** Throughout, let k be a field, and let A be a bialgebra over k with coproduct $\Delta : A \to A\otimes_k A$ and augmentation $\varepsilon : A \to k$.

Definition 2.1. An element $Q \in A$ is *grouplike* if $\Delta(Q) = Q\otimes Q$ and $\varepsilon(Q) = 1$.

Definition 2.1 is classical and standard, as are the following facts:

(1) The set of grouplike elements of A forms a monoid under multiplication, since ab is grouplike if a and b are each grouplike.
(2) If A is furthermore a Hopf algebra, then the antipode $\chi : A \to A$ yields an inverse operation on the monoid of grouplike elements of A, making that monoid into a group.
(3) A cocommutative bialgebra over an algebraically closed field (or, more generally, a pointed cocommutative bialgebra over any field) is a Hopf algebra if and only if its monoid of grouplikes forms a group. See Proposition 9.2.5 of [32] for this result, and Lemma 8.0.1 of [32] for a proof that algebraic closure of the ground field implies pointedness of the bialgebra.

There exist non-Hopf bialgebras whose grouplikes are all invertible, hence form a group: see Example 2 in [26] for an example of a cocommutative bialgebra over the real numbers which is not a Hopf algebra, but whose only grouplike is 1. Of course a non-Hopf cocommutative bialgebra cannot be primitively-generated.

Definition 2.2 is not new: it is a natural idea, and it appears, with slightly different terminology and notation, in section 1.1 of chapter II of Bourbaki's book [7]. However, Bourbaki very quickly restrict their attention to classical primitives, so the more general theory of generalized primitives does not get developed very far in [7].

Definition 2.2.

- We write $\Gamma(A)$ for the monoid of grouplike elements of A.
- Suppose $Q \in \Gamma(A)$. An element $a \in A$ is *Q-primitive* if $\Delta(a) = a \otimes Q + Q \otimes a$.
- An element $a \in A$ is a *homogeneous generalized primitive* if a is Q-primitive for some $Q \in \Gamma(A)$.
- An element $a \in A$ is a *generalized primitive* if a is a k-linear combination of homogeneous generalized primitives. We write $GP_*(A)$ for the k-vector space of generalized primitives in A.
- Suppose that the monoid $\Gamma(A)$ is commutative[8]. For each $Q \in \Gamma(A)$, we write $GP_Q(A)$ for the k-vector space of Q-primitives in A. Then

$$GP_*(A) = \bigoplus_{Q \in \Gamma(A)} GP_Q(A),$$

that is, we regard $GP_*(A)$ as a graded k-vector space, graded by the monoid $\Gamma(A)$.

2.2. **Structure of the set of generalized primitives.** In general, the commutator of two generalized primitives does not have to be a generalized primitive, so unlike classical primitives, generalized primitives in A are not guaranteed to form a sub-Lie-algebra of A under the commutator bracket! However, under some reasonable hypotheses, we *do* get that the commutator of generalized primitives is a generalized primitive, as we prove in Proposition 2.5. The relevant hypotheses are laid out in Definition 2.3:

Definition 2.3. We say that the bialgebra A has *primitive-grouplike compatibility* if, for every pair of grouplikes Q, Q' in A, every Q-primitive $a \in A$, and every Q'-primitive $a' \in A$, we have:

$$0 = aQ' \otimes Qa' + Qa' \otimes aQ' - a'Q \otimes Q'a - Q'a \otimes a'Q \in A \otimes_k A. \tag{1}$$

Example 2.4. In order of increasing generality:

- Every commutative bialgebra has primitive-grouplike compatibility.
- If every grouplike in A is contained in the center of A, then A has primitive-grouplike compatibility. When the grouplikes of a bialgebra A are contained in the center of A, we say that A *has central grouplikes.*
- If every noncentral grouplike Q in A has the property that $Qa = 0$ for every generalized primitive $a \in A$, then A has primitive-grouplike compatibility. We refer to this last condition as *strong primitive-grouplike compatibility.*

For example, the Dyer-Lashof algebra R does not have central grouplikes[9]: its noncentral grouplikes are the positive powers of Q^0. However, R *does* have strong

[8]Perhaps this assumption is not really necessary, but if we do not assume it, then we wind up talking about graded vector spaces which are graded by a *noncommutative* monoid. That is a pretty exotic kind of grading, so it seems safest to avoid it. The only reason to assume that $\Gamma(A)$ is commutative here is so that we do not have to talk about G-gradings for noncommutative G.

[9]Appendix A reviews the basic properties of the Dyer-Lashof algebra, in case this may be useful to the reader.

primitive-grouplike compatibility, since its generalized primitives are all in the two-sided ideal of elements in positive degree, and since $Q^0Q^i = 0$ and $Q^0\beta Q^i$ for all $i > 0$, by the excess relations in R.

However, R is not generated by its grouplikes and generalized primitives. For example, the element $Q^2 \in R$ is not in the subalgebra of R generated by the grouplikes and generalized primitives. We can fix this situation by filtering R by the K-adic filtration, where K is[10] the two-sided ideal $K = (Q^1, Q^2, Q^3, \dots)$ of elements in positive degree. With this filtration, the associated graded bialgebra E_0R *does* have central grouplikes. This is because, for $n > 0$, the Adem relations yield that Q^nQ^0 is a sum of products of the form Q^rQ^s for *positive* r, s. Since Q^nQ^0 is a product of an element in K-adic filtration 1 and an element of K-adic filtration 0, while each of the terms Q^rQ^s in the sum has K-adic filtration 2, we get that the product $Q^n \cdot Q^0$ is zero in the associated graded bialgebra E_0R, for $n > 0$. Meanwhile, $Q^0 \cdot Q^n$ is zero already in R, by the excess condition. So:

- $Q^0x = 0 = xQ^0$ in E_0R if x is homogeneous and of positive degree, while
- $Q^0x = xQ^0$ if x is homogeneous of degree 0, since the degree 0 subring of E_0R is simply $\mathbb{F}_p[Q^0]$.

Hence E_0R has central grouplikes.

Proposition 2.5. *Suppose that k is a field and A is a k-bialgebra whose monoid $\Gamma(A)$ of grouplikes is commutative. Then A has primitive-grouplike compatibility if and only if the generalized primitives in A are closed under the commutator bracket.*

Proof. Suppose that $a, b \in A$, and Q_a, Q_b are grouplikes in A, and a is a Q_a-primitive and b is a Q_b-primitive. Then we have

$$\begin{aligned}\Delta([a,b]) &= (a \otimes Q_a + Q_a \otimes a)(b \otimes Q_b + Q_b \otimes b) \\ &\quad - (b \otimes Q_b + Q_b \otimes b)(a \otimes Q_a + Q_a \otimes a) \\ &= aQ_b \otimes Q_a b + Q_a b \otimes aQ_b - bQ_a \otimes Q_b a - Q_b a \otimes bQ_a \\ &\quad + [a,b] \otimes Q_aQ_b + Q_aQ_b \otimes [a,b]\end{aligned}$$

so $[a, b]$ is a Q_aQ_b-primitive if and only if the equation (1), which defines primitive-grouplike compatibility, is satisfied. □

Corollary 2.6. *If A has primitive-grouplike compatibility, then $GP_*(A)$ is a sub-Lie-algebra of A, regarded as a Lie algebra via the commutator bracket. Furthermore, $GP_*(A)$ is a $\Gamma(A)$-graded Lie algebra over k. That is, given grouplikes Q, Q' in A, the Lie bracket of a Q-primitive and a Q'-primitive lies in $GP_{Q\cdot Q'}(A)$.*

Suppose that A has primitive-grouplike compatibility. Then the $\Gamma(A)$-graded Lie algebra $GP_*(A)$ has additional structure: given an element $Q \in \Gamma(A)$ and a Q'-primitive $a \in A$, we have

$$\begin{aligned}\Delta(Qa) &= (Q \otimes Q)(Q' \otimes a + a \otimes Q') \\ &= QQ' \otimes Qa + Qa \otimes QQ',\end{aligned}$$

that is, Qa is a QQ'-primitive. So the monoid $\Gamma(A)$ acts on the left on $GP_*(A)$ by k-linear endomorphisms which are grading preserving, in the sense that $Qa \in$

[10]For simplicity, the rest of this paragraph is written under the assumption that $p = 2$, but an analogous argument also works at odd primes.

$GP_{QQ'}(A)$ if $a \in GP_{Q'}(A)$. For the same reasons, $\Gamma(A)$ also has a right action on $GP_*(A)$.

If A furthermore has central grouplikes, then the left and the right actions of $\Gamma(A)$ on $GP_*(A)$ coincide, and we furthermore have

$$[Qa, b] = Q[a, b] = [a, Qb] \tag{2}$$

for any $Q \in \Gamma(A)$, i.e., the Lie bracket on $GP_*(A)$ is $k[\Gamma(A)]$-bilinear.

Evidently $GP_*(A)$ is endowed with various algebraic structures, more than simply the Lie bracket enjoyed by the classical primitives in A. This is especially true when A has central grouplikes. Our next task, accomplished in Definition-Proposition 2.9, is to say precisely what kind of structured algebraic gadget $GP_*(A)$ is.

Definition 2.7. Let G be a commutative monoid, and let k be a field. We use multiplicative notation for the monoid operation on G.

- By a *G-graded Lie algebra over k* we mean a Lie algebra L over k whose underlying k-vector space is equipped with a G-grading such that, if x, y are homogeneous elements of L such that $x \in L_m$ and $y \in L_n$, then $[x, y] \in L_{mn}$.
- By a *G-suspensive Lie algebra over k* we mean a G-graded Lie algebra L over k equipped with an action of G on L such that
  - the G-action preserves gradings, in the sense that, if $g \in G$ and x is a homogeneous element of L with $x \in L_n$, then $g \cdot x \in L_{g \cdot n}$; and
  - the Lie bracket is kG-bilinear.

  A *homomorphism of G-suspensive Lie algebras over k* is a homomorphism of the underlying Lie algebras which preserves the grading and commutes with the G-action. We denote the resulting category of G-suspensive Lie k-algebras by $\mathrm{Susp}_G \mathrm{Lie}(k)$.

When G fails to have inverses, then G-suspensive Lie algebras are much more interesting than when G is a group. In particular, there are many examples where the G-action is far from free. Consequently the results of [23] do not straightforwardly apply to such examples.

If A is a k-bialgebra with central grouplikes, then the generalized primitives in A form a $\Gamma(A)$-suspensive Lie k-algebra. A nice puzzle for the interested reader is to decide whether there is any further structure enjoyed by the generalized primitives in A. In other words:

Question 2.8. *The set of generalized primitives in A has a k-vector space structure, a Lie bracket, a G-grading, and a G-action. Is there any other natural structure on the set of generalized primitives in A?*

In Corollary 6.10 we find that the answer to Question 2.8 turns out be "no" under certain hypotheses. These hypotheses are automatically satisfied if the monoid G is a group, for example. However, in section 7 we find that, if G has noninvertible elements, then the answer to Question 2.8 is "yes": the generalized primitives in A form a very richly structured algebraic gadget. See section 7 for discussion.

Definition-Proposition 2.9. Let k be a field. Let G be a commutative monoid, and let L be a G-suspensive Lie algebra over k. Let $W(L)$ be the k-bialgebra defined as follows. As a k-algebra,

$$W(L) = (kG \otimes_k T(L)) / \left(1 \otimes (\ell\ell' - \ell'\ell) - 1 \otimes [\ell, \ell'], g \otimes \ell - 1 \otimes (g \cdot \ell)\right),$$

where $T(L)$ is the tensor k-algebra (i.e., free associative k-algebra) on L.

In order to define the coproduct on $W(L)$, it is convenient to introduce a notational novelty: we will write $\odot$ for the *internal* tensor product in $W(L)$, and we will write $\otimes$ for the *external* tensor product in $W(L) \otimes_k W(L)$. So, to specify an element of $W(L)$, we can give a k-linear combination of elements of the form $g \odot \ell$, and to specify an element of $W(L) \otimes_k W(L)$, we can give a k-linear combination of elements of the form $(g \odot \ell) \otimes (g' \odot \ell')$. With this notation in place, the coproduct on $W(L)$ is given by

$$\begin{aligned} \Delta(g \odot 1) &= (g \odot 1) \otimes (g \odot 1), & g \in G, \\ \Delta(1 \odot \ell) &= (1 \odot \ell) \otimes (Q \odot 1) + (Q \odot 1) \otimes (1 \odot \ell), & \ell \in L_Q. \end{aligned}$$

The augmentation on $W(L)$ is given by $\varepsilon(g \odot 1) = 1$ and $\varepsilon(1 \odot \ell) = 0$ for all $g \in G$ and $\ell \in L$.

We call $W(L)$ the[11] *universal G-rigid enveloping bialgebra of L.*

Proof. It is routine to check that $W(L)$ is indeed a bialgebra, as follows. One verifies that ε and Δ are k-algebra morphisms by regarding $W(L)$ as a quotient of kG tensored with the free associative k-algebra on L, and checking that Δ and ε each vanish on the relations $1 \odot (\ell\ell' - \ell'\ell) - 1 \odot [\ell, \ell']$ and $g \odot \ell - 1 \odot (g \cdot \ell)$. Counitality and coassociativity are also straightforwardly verified. □

Example 2.10. Suppose that G is the free monoid on a single generator σ. Let L be the abelian G-suspensive Lie k-algebra consisting of a single copy of k in each degree, with the action of G freely permuting the copies of k. Then:

$$\begin{aligned} UL &\cong k[x_0, x_1, x_2, \dots], \\ WL &\cong k[\sigma, x_0], \end{aligned}$$

where x_n is the element $1 \in k = L^{\sigma^n}$, the degree σ^n summand of L. The map

$$UL \to WL \tag{3}$$

sends x_n to $\sigma^n x_0$. While $x_n \in UL$ is a primitive for each n, its image $\sigma^n x_0$ in WL is a σ^n-primitive. So, while the natural map (3) is a k-algebra morphism, it does *not* preserve the coproduct. The map (3) is also neither surjective (it does not hit σ, for example) nor injective.

2.3. **Generalized-primitive generation.** It is classical that, given a Lie algebra L, its universal enveloping algebra UL is primitively-generated. However, if L is a G-suspensive Lie algebra, it is almost never the case that WL is primitively-generated. Instead, WL is easily seen to be *generalized-primitively-generated*, in the following sense:

Definition 2.11. Let A be a k-bialgebra. We say that A is *generalized-primitively-generated* if every element of A is a k-linear combination of products of grouplikes in A and generalized primitives in A.

[11]While this definition of a "universal G-rigid enveloping bialgebra" makes sense without having a definition of a "G-rigid bialgebra" in general, we do provide a definition of G-rigid bialgebras below, in Definition 4.1.

Note that being generalized-primitively-generated is a much weaker condition than being primitively-generated. For example, a bialgebra with no nonzero primitives, *and also no nonzero generalized primitives*, can still be generalized-primitively-generated, simply by being generated by grouplikes. This happens for group rings, for example.

Nevertheless, not all bialgebras are generalized-primitively-generated. For example, non-co-commutative bialgebras cannot be generalized-primitively-generated, as we now show:

Proposition 2.12. *If A is a generalized-primitively-generated k-bialgebra, then A is co-commutative.*

Proof. Straightforward: the coproduct in A is commutative on grouplikes and on generalized primitives. So if grouplikes and generalized primitives generate A, then the coproduct is commutative on all elements of A. □

The converse of Proposition 2.12 fails, however. That is, it is not the case that every co-commutative k-bialgebra is generalized-primitively-generated. An example is as follows:

Example 2.13. Let C_3 be the cyclic group of order three, and let $\mathbb{R}$ be the field of real numbers. Consider the linear dual $\mathbb{R}[C_3]^*$ of the group algebra $\mathbb{R}[C_3]$. By explicit, elementary calculation, one finds that

- the only grouplike in the Hopf algebra $\mathbb{R}[C_3]^*$ is 1,
- so the only generalized primitives in $\mathbb{R}[C_3]^*$ must be primitives,
- and there are no nonzero primitives in $\mathbb{R}[C_3]^*$,
- so the subalgebra of $\mathbb{R}[C_3]^*$ generated by grouplikes and generalized primitives is $\mathbb{R} \subseteq \mathbb{R}[C_3]^*$.

Therefore $\mathbb{R}[C_3]^*$ is not generalized-primitively-generated. The argument extends to C_n for any odd $n \geqslant 3$, because $\mathbb{R}$ is missing all odd-order roots of unity except for 1.

The bialgebra of Example 2.13 is cocommutative but not generalized-primitively-generated. However, after base change to the algebraic closure, it does become generalized-primitively-generated. This suggests the idea that perhaps cocommutative bialgebras over algebraically closed fields are generalized-primitively-generated, or more generally, that *pointed* cocommutative bialgebras are generalized-primitively-generated[12]. This would be a kind of converse to Proposition 2.12. We know no reason to expect such an idea to actually be true, however: for example, the Dyer-Lashof algebra is cocommutative but fails to be generalized-primitively-generated, even after base change to $\overline{\mathbb{F}}_p$. It is probably a very difficult problem to find a reasonable and useful sufficient condition on pointed cocommutative coalgebras which ensures that they are generalized-primitively-generated, since this begins to resemble the Andruskiewitsch-Schneider conjecture, which states that every finite-dimensional pointed Hopf algebra over an algebraically closed field of characteristic zero is generated by grouplikes and skew-primitives. See Conjecture 1.4 of [1] for the conjecture, and Theorem 5.5 of [3] for recent progress on it.

[12] See Definition 5.1 for the definition of pointedness for bialgebras and coalgebras. See Definition 2.1 for a statement of the relationship between pointedness and algebraic closure of the ground field.

3. G-suspensive vector spaces.

We continue to let G be a commutative monoid. In Definition 2.7, we defined G-suspensive Lie algebras. Recall that a G-suspensive Lie algebra is a vector space with a G-grading, a G-action, and a Lie bracket, satisfying some compatibility conditions between these three pieces of structure. In this section we make some observations about slightly weaker objects, "G-suspensive vector spaces," which have the G-grading and G-action but not the Lie bracket. See Definition 3.2 for the full definition.

First, we recall (e.g. as in Proposition 5.12 in Dwyer's chapter of [12]) the definition of the *transport category* of a monoid:

Definition 3.1. By the *transport category of* G we mean the category $\mathrm{Tr}(G)$ whose set of objects is G itself, and such that the set of morphisms $\hom_{\mathrm{Tr}(G)}(g,h)$ from g to h in $\mathrm{Tr}(G)$ is the set of elements f of G such that $fg = h$. The composite $j \circ f$ of $f \in \hom_{\mathrm{Tr}(G)}(g,h)$ with $j \in \hom_{\mathrm{Tr}(G)}(h,i)$ is defined to simply be the product jf in G.

Definition 3.2. Let k be a field. By a *G-suspensive k-vector space* we mean a functor $\phi : \mathrm{Tr}(G) \to \mathrm{Vect}(k)$.

Equivalently, a G-suspensive k-vector space consists of a G-graded k-vector space $V = \bigoplus_{g\in G} \phi(g)$ together with a unital, associative action of G, such that, if $y \in G$ and $x \in V$ is in degree $h \in G$, then $y \cdot x \in V$ is in degree yh.

Examples 3.3. For various commonly-occurring commutative monoids G, here are alternative descriptions of the category of G-suspensive k-vector spaces.

- Let $G = \langle\sigma\rangle$, the free commutative monoid on one generator σ. Then a G-suspensive k-vector space consists of a sequence $\phi(1), \phi(\sigma), \phi(\sigma^2), \dots$ of k-vector spaces together with a k-linear function
$$\sigma_{n,n+1} : \phi(\sigma^n) \to \phi(\sigma^{n+1}) \tag{4}$$
for each nonnegative integer n. (Of course there is also a morphism $\sigma_{n,m} : \phi(\sigma^n) \to \phi(\sigma^m)$ for each $n \leqslant m$, but this morphism is equal to the composite $\sigma_{m-1,m} \circ \sigma_{m-2,m-1} \circ \dots \circ \sigma_{n+1,n+2} \circ \sigma_{n,n+1}$, so it is determined by the morphisms of the type (4).)
  In other words, the category of G-suspensive k-vector spaces is equivalent to the category of sequences $V_0 \to V_1 \to \dots$ of k-vector spaces.
- Let $G = \langle\sigma \mid \sigma^2\rangle$, the cyclic group with two elements. By a similar analysis, the category of G-suspensive k-vector spaces is equivalent to the category of k-linear isomorphisms, i.e., the subcategory of the category of arrows in $\mathrm{Vect}(k)$ such that the arrow is an isomorphism.

Here is one more perspective on what "G-suspensive algebra" is about. Given a k-vector space V, it is classical that the data of a G-grading on V is equivalent to the data of a coassociative, counital map $V \to V \otimes_k kG$, i.e., the structure of a kG-comodule on V. If V is also equipped with a k-linear action of G, then given a coassociative counital map $\psi : V \to V \otimes_k kG$, we have a G-action on the domain of ψ and also the diagonal G-action on the codomain of ψ. To give the structure of a G-suspensive vector space on V is equivalent to giving a choice of coassociative counital map $\psi : V \to V \otimes_k kG$ which commutes with the G-action. So: *a G-suspensive vector space is a vector space equipped with an action of kG and a compatible coaction of kG.*

4. Rigid bialgebras.

Throughout, we continue to let G be a commutative monoid, and let k be a field.

We begin with the notion of a *G-rigid bialgebra*, i.e., a bialgebra with central grouplikes which is furthermore equipped with a choice of isomorphism between its monoid of grouplikes and G. The definition, in more detail, is as follows:

Definition 4.1. By a *G-rigid k-bialgebra* we mean a k-bialgebra A equipped with a homomorphism of k-bialgebras $\eta : kG \to A$ satisfying each of the following conditions:

- The monoid map $\Gamma(kG) \xrightarrow{\cong} \Gamma(A)$ induced by η is an isomorphism.
- The image of η lies in the center of A.

We refer to the homomorphism η as the *rigid unit map of A*. A *homomorphism of G-rigid k-bialgebras* is a homomorphism of the underlying k-bialgebras which commutes with the rigid unit maps. We denote the resulting category of G-rigid k-bialgebras by $\mathrm{Rig}_G \mathrm{Bialg}(k)$, and the full subcategory of *cocommutative* G-rigid k-bialgebras by $\mathrm{Cocomm}\,\mathrm{Rig}_G \mathrm{Bialg}(k)$.

It is worth being explicit about this point: if A is a G-rigid k-bialgebra, then the elements of G are grouplike in A, and consequently are central. Hence A is a kG-algebra. However, a G-rigid k-bialgebra is in general *not* a kG-*bi*algebra. The failure of a G-rigid k-algebra to be a kG-bialgebra is visible in the action of kG on the tensor product $A \otimes_k A$, as follows: if A is a G-rigid k-bialgebra, then for any $g \in G$ and any $a_1 \otimes a_2 \in A \otimes_k A$, we have $g \cdot (a_1 \otimes a_2) = ga_1 \otimes ga_2$. This is because g is grouplike in kG and because the rigid unit map $kG \to A$ is a bialgebra map. On the other hand, if A were instead a kG-bialgebra, then we would have $\Delta(g) = g \otimes 1 = 1 \otimes g = g(1 \otimes 1)$, which is not equal to $g \otimes g$ unless $g = 1$.

Proposition 4.2. *The functor $W : \mathrm{Susp}_G \mathrm{Lie}(k) \to \mathrm{Cocomm}\,\mathrm{Rig}_G \mathrm{Bialg}(k)$ defined in Definition-Proposition 2.9 is left adjoint to the functor*

$$GP_* : \mathrm{Cocomm}\,\mathrm{Rig}_G \mathrm{Bialg}(k) \to \mathrm{Susp}_G \mathrm{Lie}(k).$$

Proof. Suppose that L is a G-suspensive Lie k-algebra, and suppose that A is a cocommutative G-rigid k-bialgebra. It is elementary to check that the function

$$\alpha : \hom_{\mathrm{Cocomm}\,\mathrm{Rig}_G \mathrm{Bialg}(k)} (WL, A) \to \hom_{\mathrm{Susp}_G \mathrm{Lie}(k)} (L, GP_*(A))$$

is well-defined, where α is given by letting $(\alpha(f))(\ell) = f(\ell)$ for each morphism $f : WL \to A$ of G-rigid k-bialgebras. To show that GP_* is right adjoint to W, all we need is an inverse to α. Such an inverse is the function

$$\beta : \hom_{\mathrm{Susp}_G \mathrm{Lie}(k)} (L, GP_*(A)) \to \hom_{\mathrm{Cocomm}\,\mathrm{Rig}_G \mathrm{Bialg}(k)} (WL, A)$$

given on an element $f \in \hom_{\mathrm{Susp}_G \mathrm{Lie}(k)} (L, GP_*(A))$ as follows: by the universal property of the free associative k-algebra $T(L)$, there exists a unique k-algebra homomorphism $\tilde{f} : T(L) \to A$ such that $\tilde{f}(\ell) = f(\ell) \in GP_*(A) \subseteq A$ for each $\ell \in L$. Since the grouplikes in A are central, we get a well-defined k-algebra homomorphism $\overline{f} : kG \odot_k T(L) \to A$ given by $\overline{f}(g \odot \ell) = g \cdot \tilde{f}(\ell)$. (See Definition-Proposition 2.9 for the definition of the $\odot$ notation.) Since $\overline{f}(1 \odot [\ell_1, \ell_2]) = \overline{f}(1 \odot \ell_1)\overline{f}(1 \odot \ell_2) - \overline{f}(1 \odot \ell_2)\overline{f}(1 \odot \ell_1)$ and since $\overline{f}(g \odot \ell) = \overline{f}\,(1 \odot (g \cdot \ell))$, the map $\overline{f}$ factors as the projection $kG \odot_k T(L) \twoheadrightarrow W(L)$ followed by a unique kG-algebra homomorphism $W(L) \to A$, which is the desired morphism $\beta(f)$.

We need to verify that $\beta(f) : WL \to A$ is not only a morphism of kG-algebras, but in fact a morphism of G-rigid k-bialgebras. That $\beta(f)$ commutes with the G-action is straightforward from the relation $g \odot \ell = 1 \odot g \cdot \ell$ in $W(L)$. Consequently $\beta(f)$ commutes with the rigid unit maps. Since $W(L)$ is generated, as a kG-algebra, by the elements of $L \subseteq W(L)$, to check that $\beta(f)$ is a coalgebra morphism it suffices to check that $\Delta(\beta(f)(\ell)) = (\beta(f) \otimes \beta(f))\, \Delta(\ell)$ for all $\ell \in L$. This is straightforward: suppose that $\ell \in L$ is homogeneous of degree Q. Then we have:

$$\begin{aligned} \Delta(\beta(f)(\ell)) &= \Delta(f(\ell)) \\ &= f(\ell) \otimes Q + Q \otimes f(\ell) \in A \otimes_k A \\ &= (\beta(f) \otimes \beta(f))(\ell \otimes Q + Q \otimes \ell) \\ &= (\beta(f) \otimes \beta(f))(\Delta(\ell)), \end{aligned}$$

as desired.

We have $\beta(\alpha(f))(\ell) = \alpha(f)(\ell) = f(\ell)$ and $\alpha(\beta(f))(\ell) = \beta(f)(\ell) = f(\ell)$ straightforwardly from the definitions of α and β, so α and β are indeed mutually inverse. □

5. Injectivity and generalized primitives.

A classical result, Proposition 3.9 of [23], establishes that a map of augmented[13] k-coalgebras $A \to B$, with A connected and k a field, is injective if and only if its induced map $PA \to PB$ is injective. We will need an analogue of that classical result which applies to certain G-rigid bialgebras. That analogue is Proposition 5.5, and it is an easy consequence of two classical results on coalgebras and one classical result on bialgebras. We recall the three results below, as Theorems 5.2 and 5.3 and 5.4. First, we recall a classical definition (see chapter 5 of [24] for an excellent treatment of these ideas):

Definition 5.1. Let k be a field, and let A be a k-coalgebra.

- Given grouplikes g, h of A, a (g, h)*-primitive of* A is an element $x \in A$ such that $\Delta(x) = g \otimes x + x \otimes h$. An element $x \in A$ is *skew-primitive* if x is (g, h)-primitive for some grouplikes g, h of A.
- The *coradical* of A, written A_0, is the sum of all simple coalgebras of A.
- The *coradical filtration of* A is the filtration of A by subcoalgebras
$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A$$
of A, defined by letting A_n be the preimage of $A \otimes A_{n-1} + A_0 \otimes A$ under the coproduct map $A \to A \otimes_k A$.
- We say that A is *pointed* if every simple subcoalgebra of A is one-dimensional as a k-vector space. That is (see the paragraph following 5.1.5 of [24]), A is pointed if and only if the coradical of A coincides with the k-linear span of the grouplike elements of A.

Theorem 5.2. (Heyneman-Radford.) *Suppose that k is a field, and suppose that A, B are k-coalgebras. Let $f : A \to B$ be a coalgebra morphism whose restriction to A_1 is injective. Then f is injective.*

[13] To avoid possible confusion, we point out that here we are using the phrase "augmented coalgebra" in the same way as how the phrase is used in [23], i.e., a coalgebra A equipped with a suitable *unit* map $k \to A$.

Proof. The original 1974 paper of Heymeman and Radford is [14]. See Theorem 5.3.1 in [24] for a presentation (with proof) in the context of related results. □

The statement of the Taft-Wilson theorem given below, in Theorem 5.3, is only one part of the original Taft-Wilson result, but we will not need to make use of the other parts of the result in this paper.

Theorem 5.3. (Taft-Wilson.) *Suppose that k is a field, and suppose that A is a pointed k-coalgebra. Then A_1 is k-linearly spanned by the grouplikes and the skew-primitives of A.*

Proof. The original paper of Taft and Wilson, also from 1974, is [33]. It also appears, with proof, as Theorem 5.4.1 in [24], again usefully in the context of related results. □

We do not know the original reference for Theorem 5.4. We learned of it from [16], which in turn cites Lemma 1 of [27]. We do not know of an earlier reference. At least in its form for Hopf algebras rather than bialgebas, it seems to be quite well-known (e.g. "It is easy to see that any Hopf algebra generated by grouplike and skew-primitive elements is automatically pointed" in [2]).

Theorem 5.4. *Let k be a field, and let A be a k-bialgebra which is generated, as a k-algebra, by its grouplike and skew-primitive elements. Then A is pointed.*

Proof. See Corollary 1.7.3 of [16]. □

Now here is an easy consequence[14] of Theorems 5.2, 5.3, and 5.4:

Proposition 5.5. *Let k be a field, let G be a commutative monoid, and let A, B be G-rigid k-bialgebras. Suppose that A is generalized-primitively-generated. Let $f : A \to B$ be a homomorphism of G-rigid k-bialgebras. Then f is injective if and only if $GP_* f : GP_* A \to GP_* B$ is injective.*

Proof. Clearly the injectivity of f implies the injectivity of $GP_* f$. For the converse, suppose that $GP_* f$ is injective. Since A, B are G-rigid and f is a morphism of G-rigid bialgebras, f is an isomorphism on grouplikes. Since A is co-commutative (by Proposition 2.12), every skew-primitive in A must be a generalized primitive. Since A is generalized-primitively-generated, it is consequently generated by grouplikes and skew-primitives, so Theorem 5.4 implies that A is pointed. Hence, by Theorem 5.3, f is injective on A_1. Then Theorem 5.2 immediately gives us that f is injective. □

Proposition 5.5 plays an important role in the proof of Theorem 6.9, below.

6. Milnor-Moore for bialgebras in characteristic zero, torsion-free case.

Definition 6.1 recalls the Lie filtration, defined by Milnor and Moore in Definition 5.12 of [23], and offers a generalization of it.

Definition 6.1. Let k be a field.

[14]This consequence is not new: it is, for example, nearly the same as Proposition 3 of [34].

- Given a Lie k-algebra L, the *Lie filtration on* UL is the increasing filtration of UL by k-vector subspaces
$$UL \supseteq \dots \supseteq F_2UL \supseteq F_1UL \supseteq F_0UL$$
given by:
  - $F_0UL = k$, i.e, F_0UL is the image of the unit map $k \to UL$,
  - and for each positive integer n, F_nUL is the image of the multiplication map
$$L \otimes_k F_{n-1}UL \subseteq UL \otimes_k UL \to UL.$$
- Given a commutative monoid G and a G-suspensive Lie k-algebra L, by the *Lie filtration on* WL we mean the increasing filtration of WL by k-vector subspaces
$$WL \supseteq \dots \supseteq \tilde{F}_2WL \supseteq \tilde{F}_1WL \supseteq \tilde{F}_0WL$$
given by letting $\tilde{F}_nWL$ be the kG-linear span of the image of $F_nUL \subseteq UL$ in WL under the map
$$UL \to WL = (kG \otimes_k UL)/\left(g \otimes \ell - 1 \otimes (g \cdot \ell)\right).$$

Following Milnor and Moore, we write ${}^IE^0UL$ for the associated graded of the Lie filtration on UL. Similarly, we write ${}^I\tilde{E}^0WL$ for the associated graded of the Lie filtration on WL.

Observation 6.2. Here are some observations about Definition 6.1.

(1) It is classical that the elements in Lie filtration n in UL are those which are expressible as a linear combination of products of $(\leqslant n)$-tuples in $L \subseteq UL$. Similarly, the elements in Lie filtration n in WL are the linear combinations of products of grouplikes and $(\leqslant n)$-tuples in $L \subseteq WL$.

(2) It is straightforward that, if $f : L \to L'$ is a homomorphism of G-suspensive Lie k-algebras, then f is compatible with the Lie filtration, in the sense that Wf sends elements of $\tilde{F}_nWL$ to elements of $\tilde{F}_nWL'$. Consequently f induces a homomorphism ${}^I\tilde{E}^0f : {}^I\tilde{E}^0WL \to {}^I\tilde{E}^0WL'$ of associated graded bialgebras.

Let L be a Lie algebra. In section 5 of [23], Milnor and Moore define $L^\#$ to be the underlying k-vector space of L equipped with the trivial Lie bracket. Then ${}^IE^0UL$ is isomorphic to ${}^IE^0U(L^\#)$. We now consider an analogue in the suspensive setting:

Definition 6.3. Suppose we are given a commutative monoid G, a field k, and a G-suspensive Lie k-algebra L. Let $L^{\tilde{\#}}$ denote the abelian G-suspensive Lie k-algebra with the same underlying G-suspensive vector space as L. That is, $L^{\tilde{\#}}$ is the k-vector space underlying L, with the same G-grading as L, the same G-action as L, and with zero Lie bracket.

In Proposition 6.4 and elsewhere, when we deal with ${}^I\tilde{E}^0W(L)$, it pays to use terminology which clearly distinguishes the two gradings on ${}^I\tilde{E}^0W(L)$: there is a G-grading coming from the fact that L is G-suspensive, and there is a $\mathbb{N}$-grading coming from the fact that ${}^I\tilde{E}^0W(L)$ is the associated graded of the Lie filtration. Whenever there is risk of confusion, we will refer to degrees in the first grading as *suspensive degree* and degrees in the second grading as *Lie degree.*

One form (see Theorems 5.15 and 5.16 of [23]) of the classical Poincaré-Birkhoff-Witt theorem states that, if L is a Lie algebra over a field k of characteristic zero, then ${}^{I}E^0UL$ is isomorphic to the symmetric k-algebra $S_k(L^{\#})$ on the underlying k-vector space $L^{\#}$ of L. Here is the relevant G-suspensive version:

Proposition 6.4. *Let G be a commutative monoid, let k be a field of characteristic zero, and let L be a G-suspensive Lie k-algebra. Regard the underlying G-suspensive k-vector space $L^{\#}$ of L as a kG-module via the action of G on L. Equip the symmetric kG-algebra $S_{kG}(L^{\#})$ with the structure of a G-rigid k-bialgebra, by letting the elements of $L^{\#} \subseteq S_{kG}(L^{\#})$ in suspensive degree Q be Q-primitives, and letting the elements $G \subseteq kG \subseteq S_{kG}(L^{\#})$ in degree 0 be grouplike. Then we have an isomorphism of G-rigid k-bialgebras*

$$ {}^{I}\tilde{E}^0WL \cong S_{kG}(L^{\#}). $$

Proof. Clearly ${}^{I}\tilde{E}^0WL \cong {}^{I}\tilde{E}^0W(L^{\#})$ by the definition of the Lie filtration on WL. In Lie degree 0, ${}^{I}\tilde{E}^0W(L^{\#})$ is the image of the rigid unit map, while in Lie degree 1, ${}^{I}\tilde{E}^0W(L^{\#})$ is a copy of $kG \otimes_{kG} L^{\#} \cong L^{\#}$. In Lie degree $n \geqslant 1$, ${}^{I}\tilde{E}^0W(L^{\#})$ consists of the k-linear combinations of formal products of n elements of $L^{\#}$, modulo the relations enforcing multilinearity of the G-action, i.e.,

$$
\begin{aligned}
Q \cdot (\ell_1 \cdot \dots \cdot \ell_n) &= (Q\ell_1) \cdot \ell_2 \cdot \dots \cdot \ell_n && (5)\\
&= \ell_1 \cdot (Q\ell_2) \cdot \dots \cdot \ell_n && (6)\\
&= \dots \\
&= \ell_1 \cdot \ell_2 \cdot \dots \cdot (Q\ell_n). && (7)
\end{aligned}
$$

This is precisely the nth symmetric power of the kG-module $L^{\#}$. □

Corollary 6.5. *Let G be a commutative monoid, let k be a field of characteristic zero, and let L be a G-suspensive Lie k-algebra. Then the canonical map $L \to WL$ is injective.*

Proof. We have the commutative diagram of k-vector spaces

$$
\begin{array}{ccccc}
L & \longrightarrow & {}^{I}E^0UL & \longrightarrow & {}^{I}\tilde{E}^0WL \\
 & & \downarrow \cong & & \downarrow \cong \\
 & & {}^{I}E^0U(L^{\#}) & \longrightarrow & {}^{I}\tilde{E}^0W(L^{\tilde{\#}}) \\
 & & \downarrow \cong & & \downarrow \cong \\
 & & S_k(L^{\#}) & \longrightarrow & S_{kG}(L^{\tilde{\#}})
\end{array}
$$

and the composite $L \to S_{kG}(L^{\tilde{\#}})$ is an isomorphism onto the (Lie) degree 1 summand in $S_{kG}(L^{\tilde{\#}})$. So the composite $L \to {}^{I}\tilde{E}^0WL \to S_{kG}(L^{\tilde{\#}})$ is one-to-one. So $L \to {}^{I}\tilde{E}^0WL$ is one-to-one, so $L \to WL$ is one-to-one. □

Definition 6.6. Suppose we are given a commutative monoid G and a field k.

- We say that a k-linear action of G on a k-vector space V is *strictly torsion-free* if, for every $v \in V$ and every $Q \in G$ such that $Qv = 0$, we have $v = 0$.
- Suppose that V is a G-suspensive k-vector space. We say that an element $x \in V$ in degree Q is *torsion* if $Q \cdot x = 0$.

- We say that a G-suspensive k-vector space V is *torsion-free*[15] if the only torsion element of V is zero. That is, V is torsion-free if and only if, for every $Q \in G$ and every $v \in V_Q$ such that $Qv = 0$, we have $v = 0$.
- At the opposite extreme, we say that a G-suspensive k-vector space V is *torsion* if every element of V is torsion.

Note that the definition of a strictly torsion-free action of G requires only an action of G, while the definition of a torsion-free action of G requires both an action of G and a G-grading on V, i.e., a G-suspensive structure on V.

It is clear that, if a G-suspensive k-vector space is strictly torsion-free, then it is also torsion-free. It is not difficult to come up with counterexamples to the converse claim.

The action of G on a torsion-free G-suspensive k-vector space V may be far from free. The torsion-freeness condition enforces that $Q \in G$ acts faithfully on degree Q in V, but Q may act quite non-faithfully on other degrees in V. Consequently the remarks following Definition 2.7 apply here as well: torsion-free G-suspensive Lie k-algebras are not necessarily free (or projective) as kG-modules, so general results about Lie R-algebras projective over R (as in [23]) do not apply to general torsion-free G-suspensive Lie k-algebras.

Proposition 6.7. *Let G be a commutative monoid, let k be a field of characteristic zero, and let L be a* torsion-free *abelian G-suspensive Lie k-algebra. Then the natural inclusion $L \to GP_*(WL)$ is an isomorphism.*

Proof. Since L is abelian, by Proposition 6.4 we have $GP_*WL \cong GP_*S_{kG}(L)$. Choose a k-linear basis $\{x_i : i \in I\}$ for L consisting of elements homogeneous with respect to the G-grading. A homogeneous element of $S_{kG}(L)$ of Lie degree 0 is simply an element of kG. An element of $S_{kG}(L)$ of Lie degree $n > 0$ is a polynomial of degree n in the variables $\{x_i : i \in I\}$, modulo the relations (5) through (7) enforcing multilinearity of the G-action. We use the symbol $\vec{x}$ to denote such a monomial in $S_{kG}(L)$.

Similarly, an element of $S_{kG}(L) \otimes_k S_{kG}(L)$ is describable as a sum of products of elements $Q^\ell = Q \otimes 1$ and $Q^r = 1 \otimes Q$ and $\vec{x}^\ell = \vec{x} \otimes 1$ and $\vec{x}^r = 1 \otimes \vec{x}$.

From this perspective, the coproduct on $S_{kG}(L)$ sends a polynomial $f(\vec{x})$ to

$$f\left(|\vec{x}|^\ell \cdot \vec{x}^r + |\vec{x}|^r \cdot \vec{x}^\ell\right),$$

i.e., the same polynomial f but with each instance of x_i replaced by $|x_i|^\ell \cdot x_i^r + |x_i|^r \cdot x_i^\ell$, where $|x_i|$ is the suspensive degree of x_i. Consequently, if f is a Q-primitive, then

$$Q^\ell f(\vec{x}^r) + Q^r f(\vec{x}^\ell) = f\left(|\vec{x}|^\ell \cdot \vec{x}^r + |\vec{x}|^r \cdot \vec{x}^\ell\right). \tag{8}$$

Applying the multiplication map $S_{kG}(L) \otimes_k S_{kG}(L) \to S_{kG}(L)$ to (8) yields the equation

$$2Q \cdot f(\vec{x}) = f\left(2\,|\vec{x}| \cdot \vec{x}\right). \tag{9}$$

Both the coproduct and the G-action on $S_{kG}(L)$ are homogeneous with respect to the Lie grading, so if (9) is true, then it must be true of each Lie-degree-homogeneous summand on each side. In other words, if we let f_n denote the sum

[15] As far as we know, this particular notion of torsion-freeness is not the same as that studied anywhere else in the literature, including that studied in [31].

of the monomials of Lie degree n in f, then we must have

$$2Q \cdot f_n(\vec{x}) = f_n\left(2\,|\vec{x}| \cdot \vec{x}\right) \tag{10}$$

$$= 2^n\,|\vec{x}|^n\, f_n\left(\vec{x}\right). \tag{11}$$

Since the G-action on $S_{kG}(L)$ respects the G-grading on L, the only way for (10) and (11) to be satisfied is for Q to be equal to $|\vec{x}|^n$. So we finally have

$$0 = (2^n - 2)\,|\vec{x}|^n\, f_n. \tag{12}$$

Since k is a field of characteristic zero, the only way for (12) to be satisfied is for either n to be equal to 1, or for f_n to be annihilated by $|\vec{x}|^n$. But L is torsion-free, and $f_n(\vec{x})$ is in suspensive degree $|\vec{x}|^n$. So, unless $n = 1$, the only way for f_n to be annihilated by $|\vec{x}|^n$ is if $f_n = 0$.

Consequently the only nonzero Q-primitives in $W(L) \cong S_{kG}(L)$ are in Lie degree 1, i.e., they are elements of $L \subseteq WL$. □

Definition 6.8. Let G be a commutative monoid and let k be a field. A G-rigid k-bialgebra A is said to be *torsion-free* if, for every $Q \in G$ and every Q-primitive $a \in A$ such that $Q \cdot a = 0$, we have $a = 0$.

In other words, a G-rigid k-bialgebra A is torsion-free if and only if the G-suspensive k-vector space GP_*A is torsion-free.

We now introduce some abbreviations for various categories of objects we have considered in this section, and functors between them:

- We denote by $\mathrm{GPGen}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k)$ the full subcategory of $\mathrm{Cocomm}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k)$ whose objects are the generalized-primitively-generated bialgebras.
- Similarly, we denote by $\mathrm{TF}\,\mathrm{GPGen}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k)$ the full subcategory of $\mathrm{Cocomm}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k)$ whose objects are the torsion-free generalized-primitively-generated bialgebras.
- We denote by $\mathrm{TF}\,\mathrm{Susp}_G\,\mathrm{Lie}(k)$ the full subcategory of $\mathrm{Susp}_G\,\mathrm{Lie}(k)$ generated by the torsion-free G-suspensive Lie k-algebras.

Theorem 6.9. *Let k be a field of characteristic zero, and let G be a commutative monoid. Then the functors GP_* and W, with their domains and codomains restricted as follows:*

$$W : \mathrm{TF}\,\mathrm{Susp}_G\,\mathrm{Lie}(k) \to \mathrm{TF}\,\mathrm{GPGen}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k) \tag{13}$$

$$GP_* : \mathrm{TF}\,\mathrm{GPGen}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k) \to \mathrm{TF}\,\mathrm{Susp}_G\,\mathrm{Lie}(k) \tag{14}$$

are mutually inverse. Consequently the category of torsion-free G-suspensive Lie k-algebras is equivalent to the category of torsion-free generalized-primitively-generated G-rigid k-bialgebras.

Proof. Let L be a torsion-free G-suspensive Lie k-algebra. The canonical map $L \to GP_*(WL)$ is injective by Corollary 6.5. We need to show that it is also surjective. Suppose that x is a Q-primitive in WL. Passing to the associated graded of the Lie filtration on WL, we have that x represents a Q-primitive $\overline{x}$ in

$${}^{I}\tilde{E}^0(WL) = {}^{I}\tilde{E}^0 W(L^{\#}) = W(L^{\#}), \tag{15}$$

with (15) due to Proposition 6.4. By Proposition 6.7, $\overline{x}$ is in Lie filtration 1 in $W(L^{\#}) = {}^{I}\tilde{E}^0 WL$. So x is in Lie filtration 1 in WL.

Lie filtration 1 in WL consists of k-linear combinations of elements in L and elements in the image of the rigid unit map $kG \to WL$. The elements of L are certainly generalized primitives. If $x = \sum_g \alpha_g g + \sum_g \ell_g$ for some $\sum_g \alpha_g g \in kG$ and for some $\ell_g \in L_g$ for each $g \in G$, then since x is assumed to be a Q-primitive, we have

$$\sum_g (\alpha_g Q \otimes g + Q \otimes \ell_g + \alpha_g g \otimes Q + \ell_g \otimes Q) = Q \otimes x + x \otimes Q \tag{16}$$
$$= \Delta(x)$$
$$= \sum_g \alpha_g g \otimes g + \sum_g (g \otimes \ell_g + \ell_g \otimes g)\,. \tag{17}$$

Reading off the $Q \otimes Q$ terms from the left-hand side of (16) and from (17), we have the equality

$$2\alpha_Q Q \otimes Q = \alpha_Q Q \otimes Q,$$

i.e., $\alpha_Q = 0$. For any $g \in G$ such that $g \neq Q$, reading off the $g \otimes g$ terms from the left-hand side of (16) and from (17) instead yields the equality

$$0 = \alpha_g g \otimes g,$$

so $\alpha_g = 0$. So all the α coefficients are zero, i.e., x is an element of $L \subseteq GP_*(WL)$. Consequently the natural embedding $L \to GP_* WL$ is an isomorphism, i.e., GP_* is a left inverse functor to W, with domains and codomains as in (13) and (14).

We also need to show that GP_* is a right inverse functor for W, but this is straightforward: if A is a G-rigid k-bialgebra, then the image of the natural map $W(GP_* A) \to A$ is the subalgebra of A generated by the generalized primitives. So for A to be generalized-primitively-generated is precisely for the natural map $W(GP_* A) \to A$ to be surjective. Since $GP_*(W(GP_* A)) \to GP_* A$ is an isomorphism, the map $W(GP_* A) \to A$ is injective after applying GP_*. So $W(GP_* A) \to A$ is injective, by Proposition 5.5. So GP_* is both right inverse and left inverse to W, with domains and codomains as in (13) and (14). □

If G is a group, then every G-suspensive vector space is torsion-free, and every G-rigid k-bialgebra is torsion-free. Consequently, when G is a group, Theorem 6.9 reduces to:

Corollary 6.10. *Let k be a field of characteristic zero, and let G be an abelian group. Then the category of G-suspensive Lie k-algebras is equivalent to the category of generalized-primitively-generated G-rigid k-bialgebras. The equivalence is realized by the functors GP_* and W.*

Corollary 6.10 is not very remarkable. If G is an abelian group and if a k-bialgebra A is co-commutative, pointed, and G-rigid, then A is a Hopf k-algebra; see the remarks immediately following Definition 2.1. Generalized-primitively-generated rigid bialgebras are automatically pointed, by Theorem 5.4. Kostant's theorem splits every co-commutative Hopf k-algebra as the smash product of the group algebra of its grouplikes with the irreducible component of 1; see Theorem 8.1.5 of [32] for a textbook treatment. Since A is assumed to be G-rigid, the grouplikes in A are in the center of A, so this smash product in fact is simply a tensor product over k. If k furthermore has characteristic zero, then the irreducible component of 1 in A coincides with the sub-bialgebra of A generated by the primitives.

The conclusion is this: Corollary 6.10 amounts to only a rephrasing of the classical isomorphism of k-bialgebras $A \cong UPA \otimes_k kG$ that we have when G is an abelian group. Hence the only new cases of Theorem 6.9 are those in which the monoid G has noninvertible elements.

However, for topological applications, Theorem 6.9 does not go quite as far as one would like: we want to be able to handle suspensive Lie algebras and rigid bialgebras which are *not* torsion-free. For example, the associated graded bialgebra E_0R of the K-adic filtration on the Dyer-Lashof algebra is one of our motivating examples of a non-Hopf bialgebra. The bialgebra E_0R is $\mathbb{N}$-rigid, but it is very far from being torsion-free.

Omitting the adjective "torsion-free" from Theorem 6.9 results in a false claim. Here is a counterexample to the statement of Theorem 6.9 with the torsion-freeness hypothesis omitted:

Example 6.11. Let G be the free monoid generated by a single element Q. That is, G is isomorphic to $\mathbb{N}$, but we write G multiplicatively, as the monoid of nonnegative powers $\{1, Q, Q^2, Q^3, \dots\}$ of Q. Let k be a field of characteristic zero, and let L be the abelian G-suspensive Lie k-algebra which is one-dimensional as a k-vector space, concentrated in degree Q. The G-action on L is necessarily trivial. In particular, the G-suspensive Lie k-algebra L is not torsion-free.

Now consider the G-rigid k-bialgebra WL. It is isomorphic to[16]

$$(kG \odot_k UL)/\left(g \odot \ell - 1 \odot (g \cdot \ell)\right) \cong k[Q, x]/(Qx).$$

Of course x is Q-primitive in WL. But consider x^2:

$$\begin{aligned}\Delta(x^2) &= x^2 \otimes Q^2 + 2Qx \otimes Qx + Q^2 \otimes x^2 \\ &= x^2 \otimes Q^2 + Q^2 \otimes x^2,\end{aligned}$$

so x^2 is a Q^2-primitive in WL. A similar argument shows that all the positive powers of x in WL are generalized primitives. So GP_*WL is an infinite-dimensional G-suspensive Lie k-algebra, and $L \to GP_*WL$ is not surjective. So GP_* and W can fail to be mutually inverse, if we allow non-torsion-free G-suspensive Lie algebras.

In the next section we consider what additional structure on suspensive Lie algebras one would have keep track of, in order to have a completely general analogue of Theorem 6.9 which would apply to *all* generalized-primitively-generated rigid bialgebras.

7. The prospects for a completely general Milnor-Moore theorem for bialgebras.

In Example 6.11, we saw a G-suspensive Lie algebra whose canonical map $L \to GP_*(WL)$ failed to be surjective. This failure of surjectivity came about because L contained a torsion element, i.e., an element $x \in L_Q$ such that $Q \cdot x = 0$.

Perhaps it is becoming clear that, in a rigid bialgebra A which isn't torsion-free, the collection of generalized primitives is richly structured:

- If $x \in GP_Q(A)$ and $Qx = 0$, then x^2 is in $GP_{Q^2}(A)$. So there is a squaring operation on the torsion elements of $GP_*(A)$.
- More generally, if $x_1 \in GP_{Q_1}(A)$ and $x_2 \in GP_{Q_2}(A)$ and $Q_1x_2 = 0$, then $x_1x_2 \in GP_{Q_1Q_2}(A)$.

[16]The notation $\odot$ was defined in Definition-Proposition 2.9.

- More generally, if $x_1 \in GP_{Q_1}(A)$ and $x_2 \in GP_{Q_2}(A)$ and $Q_1x_2 \otimes Q_2x_1 + Q_2x_1 \otimes Q_1x_2 = 0$ in $A \otimes_k A$, then $x_1x_2 \in GP_{Q_1Q_2}(A)$. So there is a product operation on $GP_*(A)$ defined on those pairs (x_1, x_2) on which the operation
$$s_2(x_1 \otimes x_2) = |x_1|\, x_2 \otimes |x_2|\, x_1 + |x_2|\, x_1 \otimes |x_1|\, x_2$$
vanishes, where $|x|$ denotes the degree of x in $GP_*(A)$.
- More generally, we have a n-ary product operation defined on n-tuples $(x_1, \dots, x_n)$ of elements in GP_*A which are in the kernel of the k-linear function $s_n : L^{\otimes_k n} \to L^{\otimes_k 2}$ defined as follows:
  - Given a subset $U \subseteq \{1, \dots, n\}$, an element $i \in \{1, \dots, n\}$, and an element $\ell \in L$, write $x(U, \ell, i)$ for the element of WL given by letting $x(U, \ell, i)$ be $|\ell|$ if $i \in U$, and letting $x(U, \ell, i)$ be ℓ if $i \notin U$.
  - Let $s_n(\ell_1 \otimes \cdots \otimes \ell_n)$ be the element
$$\sum_{U \subseteq \{1,\dots,n\}: 0<|U|<n} \left(\prod_{i=1}^{n} x(U, \ell, i) \right) \otimes \left(\prod_{i=1}^{n} x(U', \ell, i) \right)$$
of $WL \otimes_k WL$, where U' is the complement of $U \subseteq \{1, \dots, n\}$.

  If $(x_1, \dots, x_n) \in \ker s_n$, then $x_1 \cdot \cdots \cdot x_n$ is a generalized primitive in A.

The reason that these strange functions s_n arise is that, if $x_1, \dots, x_n \in A$ have the property that x_i is a $|x_i|$-primitive for each i, then $s_n(x_1 \otimes \cdots \otimes x_n)$ is equal to the difference
$$\Delta(x_1 \cdots x_n) - |x_1 \cdots x_n| \otimes (x_1 \cdots x_n) - (x_1 \cdots x_n) \otimes |x_1 \cdots x_n|\,.$$
Hence the vanishing of $s_n(x_1 \otimes \cdots \otimes x_n)$ is equivalent to the product of the generalized primitives $x_1, \dots, x_n$ also being a generalized primitive.

To give a full theory of structured Lie algebras which is equivalent (over a field of characteristic zero) to generalized-primitively-generated rigid bialgebras, those Lie algebras would need to be equipped with a large amount of structure: for each element $x \in \ker s_n \subseteq WL^{\otimes_k n}$, we would need to record a product element $m_n(x) \in L$. We could then impose the relation $m_n(x) - x$ on WL to get a quotient bialgebra of WL in which the "spurious" (i.e., not in the image of $L \hookrightarrow WL$) generalized primitives in WL are identified with elements in Lie filtration 1, i.e., elements in the image of $L \hookrightarrow WL$.

But this is a tall order. It means keeping track of an n-ary operation m_n on L for each $n \geqslant 2$. It would not be enough to confine our attention to the case $n = 2$, i.e., the generalized primitives in WL which are linear combinations of products of pairs of elements in L. Recording only the data of such linear combinations of products of pairs would only enable us to impose the correct relations on WL to quotient out the generalized primitives in Lie filtration 2. To capture the "spurious" generalized primitives in WL which are in Lie filtration n, we need the data encoded by the n-ary product operation $m_n : \ker s_n \to L$.

Recording the data of these n-ary product operations $m_2, m_3, \dots$, and axiomatizing the various properties and compatibilities that they satisfy, involves some ugly bookkeeping. While the authors worked out some of the resulting theory and surmise that it can be made to work, we do not feel that it winds up being valuable, because the resulting structured Lie algebras are such a headache to use that one is better off just working with the bialgebras.

This is made clear already in one of our motivating examples, the associated graded bialgebra E_0R of the K-adic filtration on the Dyer-Lashof algebra R, where K is the ideal of R generated by all homogeneous elements in positive degree. For simplicity, consider the case $p = 2$. Then $Q^0Q^n = 0 = Q^nQ^0$ for all $n > 0$, and consequently $\Delta(x) = (Q^0)^n \otimes x + x \otimes (Q^0)^n$ for all x in internal degree $n > 0$ in E_0R. In other words, *every homogeneous element of positive internal degree in E_0R is a generalized primitive.* Furthermore, the vanishing of Q^0x for all x in positive degree means that the operations $s_2, s_3, \dots$ vanish completely on elements of E_0R in positive suspensive degrees. Consequently, to use the theory suggested in the previous paragraph, the products $m_2, m_3, \dots$ would need to record the full data of *all* of the products of homogeneous elements of positive internal degree in E_0R. This is a ridiculous situation: rather than the Lie algebra of generalized primitives offering a simpler and more familiar algebraic structure than the bialgebra, we find ourselves needing to supplement the Lie bracket on E_0R with essentially the full structure of all multiplications on E_0R, encoded in an unfamiliar way! We might as well have stuck with E_0R itself.

Our conclusion is that we do not have much confidence in the possibility of a *useful* Milnor-Moore theorem for *arbitrary* rigid bialgebras over a field of characteristic zero. However, for *some* families of rigid bialgebras (e.g. the torsion-free rigid bialgebras), we saw already in Theorem 6.9 that a useful and satisfying Milnor-Moore theorem can indeed be obtained. So our next task is to formulate and prove a useful Milnor-Moore theorem for a family of bialgebras including E_0R and others which structurally resemble E_0R.

8. Milnor-Moore for left-sided bialgebras in characteristic zero.

The associated graded bialgebra E_0R of the $(Q^1, Q^2, \dots)$-adic filtration on the Dyer-Lashof algebra R is a generalized-primitively-generated $\mathbb{N}$-rigid bialgebra, but since it is not torsion-free, Theorem 6.9 does not apply to it. Indeed, *every* generalized primitive in E_0R is torsion[17].

Despite its extreme failure to be torsion-free, E_0R has a special property which makes it much better suited to a Lie-algebra-theoretic analysis than many other non-torsion-free rigid $\mathbb{N}$-bialgebras. The essential property is the following:

Proposition 8.1. *Let $x, y \in E_0R$ be homogeneous elements of internal degrees $|x|$ and $|y|$, respectively. Suppose $|x|$ and $|y|$ are each positive.*

- *If $p > 2$ and $|x| \leqslant |y|$, then $xy = 0$.*
- *If $p = 2$ and $|x| < |y|$, then $xy = 0$.*

Proof. This is simply a consequence of the fact that all monomials of negative excess are zero in R; see appendix A. □

Definition 8.2. Let G be a commutative monoid.

- Given $Q, Q' \in G$, we say that Q *divides* Q' if there exists some $g \in G$ such that $gQ = Q'$.
- Suppose that k is a field and that A is a G-rigid k-bialgebra. We say that A is *left-sided* if, for all $Q, Q' \in G$ such that Q divides Q', for all $x \in GP_Q(A)$ and all $y \in GP_{Q'}(A)$ we have $xy = 0$.

[17] See Definition 6.6 for the definition of torsion elements in suspensive Lie algebras.

- A *morphism of left-sided G-rigid k-bialgebras* is simply a morphism of G-rigid k-bialgebras whose domain and codomain are each left-sided. We write $\mathrm{LeftSided}\,\mathrm{Rig}_G\,\mathrm{Bialg}(k)$ for the resulting category of left-sided G-rigid k-bialgebras.

The advantage of working with left-sided bialgebras is that, even when they fail to be torsion-free (in the sense of Definition 6.8), there is no need to bring in the arcane product structure $m_2, m_3, \dots$ from section 7. Use of the products $m_2, m_3, \dots$ is unnecessary since in a left-sided bialgebra, the product of two (or finitely many) generalized primitives is describable entirely in terms of the Lie bracket. That is,

$$xy = \begin{cases} [x,y] & \text{if } |x| > |y| \\ -[y,x] & \text{otherwise.} \end{cases}$$

As a consequence of Proposition 8.1, E_0R is left-sided for primes $p > 2$. If $p = 2$, then we have some nonzero products of elements in E_0R of the same internal degree (e.g. Q^1Q^1). So while E_0R is not left-sided for $p = 2$, all such nonzero products arise from the squaring operation in E_0R. Consequently the full product structure of E_0R is recoverable even when $p = 2$ by a combination of the Lie bracket and the restriction map on the generalized primitives. The presence of the restriction is a feature of generalized primitives in bialgebras over fields of positive characteristic. Ultimately we would like to develop the theory of the present paper (including left-sided bialgebras) in positive characteristic, for the sake of understanding Lie-algebraic aspects of the Dyer-Lashof algebra. However, in this paper we confine ourselves to the characteristic zero case, where there is already much to be said.

The definition of left-sidedness for a rigid bialgebra A is a condition on the vanishing of certain products of generalized primitives. This condition also has consequences for other products. For example, Proposition 8.3 establishes that certain products of grouplikes and generalized primitives vanish as well:

Proposition 8.3. *Let G be a commutative monoid, and let k be a field of characteristic not equal to 2. Then the following are each true:*

(1) *In a left-sided G-rigid k-bialgebra, every generalized primitive is torsion.*
(2) *If V is a torsion G-suspensive k-vector space, then $V_Q = 0$ for each element $Q \in G$ which has an inverse.*

Proof.

(1) Let $Q \in G$, and let $x \in A$ be a Q-primitive. We have $x^2 = 0$ by the definition of left-sidedness. Now apply the coproduct:

$$\begin{aligned} 0 &= \Delta(x^2) \\ &= (Q \otimes x + x \otimes Q)^2 \\ &= Q^2 \otimes x^2 + x^2 \otimes Q^2 + 2Qx \otimes Qx \\ &= 2Qx \otimes Qx. \end{aligned}$$

Since k is a field of characteristic not 2, the only way for an element $a \in A$ to have the property that $2a \otimes a \in A \otimes A$ is zero is for a itself to be zero. So we must have $Qx = 0$.

(2) If $Q \in G$ has an inverse Q^{-1}, then for each $x \in V_Q$, we have $x = Q^{-1}(Qx) = Q^{-1} \cdot 0 = 0$.

□

Definition 8.4. Let k be a field, let G be a commutative monoid, and let L be a G-suspensive Lie k-bialgebra. By the *universal left-sided G-rigid enveloping algebra of L,* written ZL, we mean the k-algebra which is the quotient of WL by the two-sided ideal I generated by:

- all products of the form $\ell_1\ell_2$, where $\ell_1 \in GP_{Q_1}(WL)$ and $\ell_2 \in GP_{Q_2}(WL)$ and Q_1 divides Q_2 in G,
- and all products of the form $Q\ell$, where $Q \in G$ and $\ell \in GP_Q(WL)$.

While the vanishing of the products $\ell_1\ell_2$ in Definition 8.4 is clearly necessary in order for ZL to be a left-sided bialgebra, the vanishing of the products $Q\ell$ is also necessary, due to Proposition 8.3. (Specifically, if we fail to include the products $Q\ell$ in the ideal I, then the resulting ideal would fail to be a bi-ideal, so the resulting quotient ZL/I would fail to be a bialgebra.)

Proposition 8.5. *Let k be a field, let G be a commutative monoid, and let L be a G-suspensive Lie k-bialgebra. The two-sided ideal I of WL, defined in Definition 8.4, is a bi-ideal in WL. Consequently ZL is a k-bialgebra, and $WL \to ZL$ is a surjective k-bialgebra morphism.*

Proof. For I to be a bi-ideal, it must satisfy $\Delta(I) \subseteq I \otimes WL + WL \otimes I$ and $\varepsilon(I) = 0$. Suppose $\ell_1 \in GP_{Q_1}(WL)$ and $\ell_2 \in GP_{Q_2}(WL)$ and Q_1 divides Q_2 in G. Then there exists a $g \in G$ such that $Q_2 = gQ_1$. Consequently we have

$$\begin{aligned}\Delta(\ell_1\ell_2) &= \Delta(\ell_1)\Delta(\ell_2) \\ &= \ell_1\ell_2 \otimes Q_1gQ_1 + \ell_1gQ_1 \otimes Q_1\ell_2 + Q_1\ell_2 \otimes \ell_1gQ_1 + Q_1gQ_1 \otimes \ell_1\ell_2 \\ &\subseteq I \otimes WL + WL \otimes I,\end{aligned}$$

since $\ell_1\ell_2$ and $Q_1\ell_1$ are each in I. Similarly, if $\ell \in L_Q$, then:

$$\begin{aligned}\Delta(Q\ell) &= Q\ell \otimes Q^2 + Q^2 \otimes Q\ell \\ &\subseteq I \otimes WL + WL \otimes I,\end{aligned}$$

since $Q\ell \in I$. So $\Delta(I) \subseteq WL \otimes I + I \otimes WL$.

As for the augmentation, since $\varepsilon(\ell) = 0$ for all $\ell \in L_Q$, and since ε is an algebra homomorphism, we have that ε vanishes on the generators of the ideal I, hence on all elements of I, as desired. □

Since ZL is a quotient of WL by elements contained in positive Lie filtration, the map $WL \to ZL$ is an isomorphism in Lie filtration 0, i.e., an isomorphism on grouplikes. So, since WL is G-rigid, the k-bialgebra ZL is also G-rigid. In fact, under a certain reasonable hypothesis on the monoid G, the bialgebra ZL enjoys much stronger properties, proven below in Proposition 8.8.

The associated graded bialgebra E_0R of the K-adic filtration on the Dyer-Lashof algebra R is $\mathbb{N}$-rigid. The commutative monoid $\mathbb{N}$ has the property that its divisibility ordering is total. That is, given two elements Q, Q' of $\mathbb{N}$, either Q divides Q' or Q' divides Q. (This statement looks strange because of course it is not true that, given two nonnegative integers, one is necessarily a divisor of the other. This strangeness is an artifact of mixing additive and multiplicative notation. For the sake of the terminology of Definition 8.2, it is better to regard $\mathbb{N}$ as the free monoid on one generator, and to use multiplicative notation for it.) Hence $\mathbb{N}$ is *linear* in the following sense:

Definition 8.6. We say that a commutative monoid G is *linear* if, for each pair $Q, Q' \in G$, either Q divides Q' or Q' divides Q.

Lemma 8.7. *Let k, G, L be as in Proposition 8.5. Suppose that G is linear. Then the map of Lie algebras $GP_*(WL) \to GP_*(ZL)$ is surjective.*

Proof. By the linearity of G, given any pair of elements ℓ_1, ℓ_2 in Lie degree 1 in WL, either the degree of ℓ_1 divides the degree of ℓ_2, or vice versa. Without loss of generality, assume that the degree of ℓ_1 divides that of ℓ_2. Then, in the quotient ZL of WL, we have $\ell_1\ell_2 = 0$ and consequently $\ell_2\ell_1 = [\ell_2, \ell_1]$. Hence, in ZL, a product of elements in Lie filtration 1 is also in Lie filtration 1, i.e., the Lie filtration of ZL collapses after the first stage. Hence every element of ZL is a linear combination of grouplikes and grouplikes multiplied by elements of L. In particular, the generalized primitives of ZL—those elements of Lie filtration 1 which do not lie in Lie filtration 0—are in the image of the composite map $L \to GP_*(WL) \to GP_*(ZL)$, hence in the image of the map $GP_*(WL) \to GP_*(ZL)$. □

Proposition 8.8. *Let k, G, L be as in Proposition 8.5. Suppose that G is linear. Then the G-rigid k-bialgebra ZL is left-sided and generalized-primitively-generated.*

Proof. Let $Q, Q' \in G$, suppose that Q divides Q', and suppose that $x \in GP_Q(ZL)$ and $y \in GP_{Q'}(ZL)$. Use Lemma 8.7 to lift x to an element $\tilde{x} \in GP_Q(WL)$, and lift y to an element $\tilde{y} \in GP_{Q'}(WL)$. Then $\tilde{x}\tilde{y}$ is in the kernel of $WL \to ZL$. So $xy = 0$, i.e., ZL is left-sided.

As for the claim of generalized-primitive generation: since WL is generalized-primitively-generated, every quotient bialgebra of WL is generalized-primitively-generated. In particular, ZL is generalized-primitively-generated. □

Theorem 8.9. *Let G be a* linear *commutative monoid, and let k be a field of characteristic zero. Write* $\mathrm{TSusp}_G \mathrm{Lie}(k)$ *for the category of torsion G-suspensive Lie k-algebras, and write* $\mathrm{LeftSided\,GPGen\,Rig}_G \mathrm{Bialg}(k)$ *for the category of left-sided generalized-primitively-generated G-rigid k-bialgebras. Then the functors*

$$GP_* : \mathrm{LeftSided\,GPGen\,Rig}_G \mathrm{Bialg}(k) \to \mathrm{TSusp}_G \mathrm{Lie}(k) \quad \textit{and}$$
$$Z : \mathrm{TSusp}_G \mathrm{Lie}(k) \to \mathrm{LeftSided\,GPGen\,Rig}_G \mathrm{Bialg}(k)$$

are mutually inverse. Consequently the category of torsion G-suspensive Lie k-algebras is equivalent to the category of left-sided generalized-primitively-generated G-rigid k-bialgebras.

Proof. Given a G-suspensive Lie k-algebra L, we have the composite map of k-bialgebras $L \hookrightarrow GP_*(WL) \to GP_*(ZL)$. Write f for this composite map $L \to GP_*(ZL)$. If L is torsion, then we claim that f is an isomorphism. The proof is as follows:

Injectivity of f: We have the commutative diagram of k-vector spaces

$$\begin{array}{ccccccc}
L & \longrightarrow & {}^I E^0 UL & \longrightarrow & {}^I \tilde{E}^0 WL & \longrightarrow & {}^I \tilde{E}^0 ZL \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
 & & {}^I E^0 U(L^{\#}) & \longrightarrow & {}^I \tilde{E}^0 W(L^{\tilde{\#}}) & \longrightarrow & {}^I \tilde{E}^0 Z(L^{\tilde{\#}}) \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & & S_k(L^{\#}) & \longrightarrow & S_{kG}(L^{\tilde{\#}}) & \longrightarrow & kG \oplus L^{\tilde{\#}}.
\end{array}$$

The composite $L \to kG \oplus L^{\tilde{\#}}$ is an isomorphism of k-vector spaces onto Lie degree 1 in $kG \oplus L^{\tilde{\#}}$. So $L \to {}^I\tilde{E}^0 ZL$ composed with a map ${}^I\tilde{E}^0 ZL \to kG \oplus L^{\tilde{\#}}$ is injective, so $L \to {}^I\tilde{E}^0 ZL$ is injective, and consequently $L \to ZL$ is injective.

Surjectivity of f: This is a consequence of the argument given in the proof of Lemma 8.7.

So $Z : \mathrm{Susp}_G \mathrm{Lie}(k) \to \mathrm{LeftSided\,GPGen\,Rig}_G \mathrm{Bialg}(k)$ is right inverse to GP_*.

Now suppose that A is a left-sided G-rigid k-bialgebra. Then $Z(GP_*A)$ is the free left-sided G-rigid k-bialgebra on the G-suspensive Lie k-algebra GP_*A, so we have a canonical map $g : Z(GP_*A) \to A$. If A is furthermore assumed to be generalized-primitively-generated, then $Z(GP_*A)$ is the free left-sided rigid bialgebra on a set of generators for A, so $g : Z(GP_*A) \to A$ is surjective. Since $GP_*(g) : GP_*(Z(GP_*A)) \to GP_*(A)$ is an isomorphism by the previous part of this theorem, it is in particular injective, so by Proposition 5.5, g itself is injective. So g is an isomorphism. So $Z : \mathrm{Susp}_G \mathrm{Lie}(k) \to \mathrm{LeftSided\,GPGen\,Rig}_G \mathrm{Bialg}(k)$ is also left inverse to GP_*. □

Appendix A. Review of the Dyer-Lashof algebra.

The material in this appendix is classical; see Theorem 1.1 of Steinberger's chapter "Homology operations for H_∞ and H_n ring spectra" in [8], or the first chapter of [11], for example.

We recall a presentation for the Dyer-Lashof algebra. If $p = 2$, let $R(-\infty)$ denote the free associative graded $\mathbb{F}_2$-algebra on generators $Q^0, Q^1, Q^2, \dots,$ with Q^i in grading degree i, modulo the Adem relation

$$Q^r Q^s = \sum_i \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i$$

for all $r > 2s$.

For an odd prime p, let $R(-\infty)$ denote instead the free associative graded $\mathbb{F}_p$ algebra on generators $Q^0, Q^1, Q^2, \dots$ and $\beta Q^0, \beta Q^1, \beta Q^2, \dots$ with Q^i in grading degree $2i(p-1)$ and βQ^i in grading degree $2i(p-1)-1$, modulo the Adem relations

$$\begin{aligned} Q^r Q^s &= \sum_i (-1)^{r+i} \binom{pi-(p-1)s-i-1}{pi-r} Q^{r+s-i} Q^i, \\ Q^r \beta Q^s &= \sum_i (-1)^{r+i} \binom{pi-(p-1)s-i}{pi-r} \beta Q^{r+s-i} Q^i \\ &\quad - \sum_i (-1)^{r+i} \binom{pi-(p-1)s-i-1}{pi-r-1} Q^{r+s-i} \beta Q^i, \\ \beta Q^r \beta Q^s &= -\sum_i (-1)^{r+i} \binom{pi-(p-1)s-i-1}{pi-r-1} \beta Q^{r+s-i} \beta Q^i. \end{aligned}$$

for all $r > ps$.

When $p = 2$, a useful notational convention which is sometimes used (e.g. in [15]) is to write

- Q^r rather than Q^{2r} for the generator for $R(-\infty)$ in degree $2r$, and
- βQ^r rather than Q^{2r-1} for the generator for $R(-\infty)$ in degree $2r-1$.

With these conventions, the Adem relations and degrees for the generators of $R(-\infty)$ at the prime 2 are the same as the Adem relations and degrees for the generators of $R(-\infty)$ at odd primes.

Let $\beta^{\varepsilon_1} Q^{i_1} \beta^{\varepsilon_2} Q^{i_2} \cdots \beta^{\varepsilon_d} Q^{i_d}$ be a monomial in this free associative graded algebra. The *excess* of this monomial is defined to be $i_1 - \sum_{j=2}^d i_j$ when $p = 2$ and

$$2i_1 - \varepsilon_1 - \sum_{j=2}^{d} \left(2i_j(p-1) - \varepsilon_j\right)$$

when p is odd. For an integer e, let J_e be the two-sided ideal of $R(-\infty)$ generated by all monomials of excess $< e$. We let $R(e) = R(-\infty)/J_e$. The special case $e = 0$ is called the *Dyer-Lashof algebra*. We often write R for $R(0)$.

The coproduct and augmentation on R are given by

$$\begin{aligned} \Delta(Q^n) &= \sum_{j=0}^{n} Q^j \otimes Q^{n-j}, \\ \Delta(\beta Q^{n+1}) &= \sum_{j=0}^{n} \left(\beta Q^{j+1} \otimes Q^{n-j} + \beta Q^j \otimes Q^{n+1-j}\right), \\ \varepsilon(Q^0) &= 1, \\ \varepsilon(Q^n) &= 0 \quad \text{if } n > 0. \end{aligned}$$

A nice reference for the coproduct is Theorem 2.3 in the first chapter of [11]. Since Q^0 is a grouplike but not invertible, one sees immediately that R cannot be a Hopf algebra.

References

[1] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Finite quantum groups and Cartan matrices. *Adv. Math.*, 154(1):1–45, 2000.

[2] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Pointed Hopf algebras. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 1–68. Cambridge Univ. Press, Cambridge, 2002.

[3] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. On the classification of finite-dimensional pointed Hopf algebras. *Ann. of Math. (2)*, 171(1):375–417, 2010.

[4] Alessandro Ardizzoni. A Milnor-Moore type theorem for primitively generated braided bialgebras. *J. Algebra*, 327:337–365, 2011.

[5] Alessandro Ardizzoni and Claudia Menini. Milnor-Moore categories and monadic decomposition. *J. Algebra*, 448:488–563, 2016.

[6] M. Basterra. André-Quillen cohomology of commutative S-algebras. *J. Pure Appl. Algebra*, 144(2):111–143, 1999.

[7] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[8] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[9] Henri Cartan and Jean-Pierre Serre. Espaces fibrés et groupes d'homotopie. I. Constructions générales. *C. R. Acad. Sci. Paris*, 234:288–290, 1952.

[10] Frédéric Chapoton. Un théorème de Cartier-Milnor-Moore-Quillen pour les bigèbres dendriformes et les algèbres braces. *J. Pure Appl. Algebra*, 168(1):1–18, 2002.

[11] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[12] William G. Dwyer and Hans-Werner Henn. *Homotopy theoretic methods in group cohomology.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001.

[13] François Goichot. Un théorème de Milnor-Moore pour les algèbres de Leibniz. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 111–133. Springer, Berlin, 2001.

[14] Robert G. Heyneman and David E. Radford. Reflexivity and coalgebras of finite type. *J. Algebra*, 28:215–246, 1974.

[15] David Kraines and Thomas Lada. The cohomology of the Dyer-Lashof algebra. In *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of *Contemp. Math.*, pages 145–152. Amer. Math. Soc., Providence, RI, 1983.

[16] Larry A. Lambe and David E. Radford. *Introduction to the quantum Yang-Baxter equation and quantum groups: an algebraic approach*, volume 423 of *Mathematics and its Applications.* Kluwer Academic Publishers, Dordrecht, 1997.

[17] Jean-Louis Loday. Generalized bialgebras and triples of operads. *Astérisque*, (320):x+116, 2008.

[18] J. P. May. The cohomology of restricted Lie algebras and of Hopf algebras. *J. Algebra*, 3:123–146, 1966.

[19] J. P. May and K. Ponto. *More concise algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories.

[20] J. Peter May. *The cohomology of restricted Lie algebras and of Hopf algebras: application to the Steenrod algebra.* ProQuest LLC, Ann Arbor, MI, 1964. Thesis (Ph.D.)–Princeton University.

[21] J. Peter May. The cohomology of restricted Lie algebras and of Hopf algebras. *Bull. Amer. Math. Soc.*, 71:372–377, 1965.

[22] Haynes Miller. A spectral sequence for the homology of an infinite delooping. *Pacific J. Math.*, 79(1):139–155, 1978.

[23] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[24] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[25] Stewart B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.

[26] David E. Radford. On bialgebras which are simple Hopf modules. *Proc. Amer. Math. Soc.*, 80(4):563–568, 1980.

[27] David E. Radford. Irreducible representations of $U_q(g)$ arising from $\mathrm{Mod}^{\bullet}_{C^{1/2}}$. In *Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992)*, volume 7 of *Israel Math. Conf. Proc.*, pages 143–170. Bar-Ilan Univ., Ramat Gan, 1993.

[28] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.

[29] María Ronco. A Milnor-Moore theorem for dendriform Hopf algebras. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(2):109–114, 2001.

[30] María Ronco. Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras. *J. Algebra*, 254(1):152–172, 2002.

[31] Jonathan A. Scott. A torsion-free Milnor-Moore theorem. *J. London Math. Soc. (2)*, 67(3):805–816, 2003.

[32] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[33] Earl J. Taft and Robert Lee Wilson. On antipodes in pointed Hopf algebras. *J. Algebra*, 29:27–32, 1974.

[34] Hiroshi Yanagihara. On homomorphisms of cocommutative coalgebras and Hopf algebras. *Hiroshima Math. J.*, 17(2):433–446, 1987.

Email address: `joeybf@wayne.edu`
Email address: `yatin@wayne.edu`
Email address: `asalch@wayne.edu`

Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202