Foundations of Mathematics, Lecture 5

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Cardinality

1. For finete sets, we just count the elements
2. We make some general observations in the finite case:
3. A subset cannot be larger, a proper subset must be smaller
4. We can make an injective mapping from smaller to larger-or-equal sets, and a surjective mapping from larger to smaller-or-equal
5. We can make a bijective mapping exactly when the two sets have the same size
6. We use some of these observations to *define* cardinality for infinite sets

Not all the above stays true in the infinite case

- A set can have the same cardinality as a proper subset: for example there are as many numbers as there are even numbers
- Also, as many even numbers as odd numbers
- Cardinality of the natural numbers is called $\aleph_0$
- Lots of things have this cardinality: integers, rationals, algebraic numbers, computable numbers, all finite subsets of the integers,...
- But not all subsets of the integers, 2^S is always strictly greater than S

The main theorems

- Cantor's Theorem: sets are strictly smaller than their powersets $|S| < |2^S|$
- Bernstein-Schröder Theorem: two injections make a bijection HW 5.1 Prove this with "closed book" (don't use wikipedia or textbooks, make your own effort)
- $|\mathbb{R}| > \aleph_0$
- Independence of Continuum Hypothesis (Gödel 1931 + Cohen 1963)

Existence proof by cardinality argument

1. Important theorem: $\mathfrak{c} > \aleph_0$ Proof: by diagonalization
2. If a set X has cardinality $> \aleph_0$, and a property P is enjoyed only by denumerably many members of X, it follows that *there must be* elements h of X for which $P(h)$ is false
3. Example: let X be $\mathbb{R}$, and P be $\exists p, q \in \mathbb{Z} : x = p/q$ (i.e. $P(h)$ means 'h is rational').
4. Using the above we can *prove* the existence of irrational numbers, but we cannot *construct* one
5. We know how to construct irrationals e.g. $\sqrt{2}$ The classic proof of $\sqrt{2} \neq p/q$ is by "minimum counterexamle" see CPZ 6.4
6. But often we have pure existence proofs that offer no construction
7. HF5.2-7 = CPZ 11.20,22,24,32,46