# Foundations of Mathematics, Lecture 2

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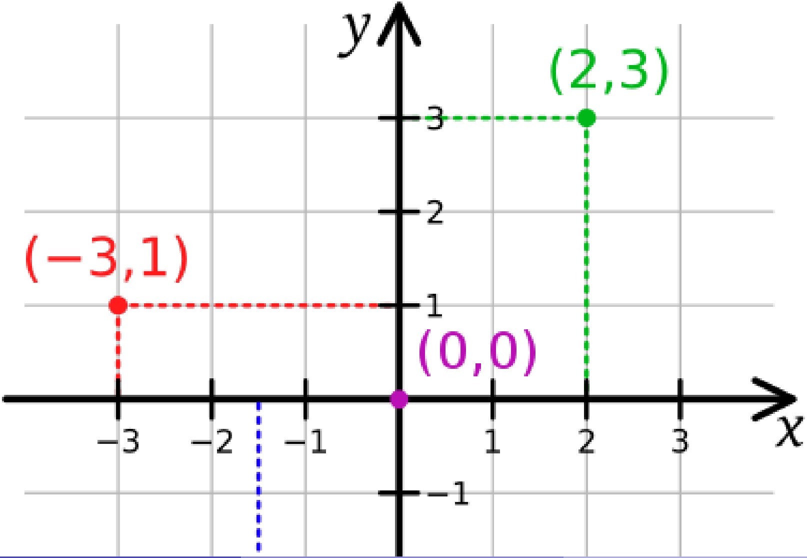
# We will build

- Ordered pairs from unordered pairs
- Relations from ordered pairs
- Functions from relations
- Operations from functions
- Structures from operations
- Model structures from structures
- Semantics from model structures

# Ordered pairs

- In sets, $\{1,2\} = \{2,1\}$. We need *ordered pairs*, where $(1,2) \neq (2,1)$
- One approach (Haussdorf 1914): number the coordinates: define $(x,y)$ by $\{\{x,1\},\{y,2\}\}$. This gets trickier when $x, y$ are numbers, and at any rate it presupposes the idea of a number
- The standard approach (Kuratowski 1921) $(x,y) \stackrel{\text{def}}{=} \{\{x\},\{x,y\}\}$
- Variants are possible, don't bring much novelty, see https://en.wikipedia.org/wiki/Ordered_pair
- Now we can build the cartesian or *direct* product $A \times B$ of two sets $A$ and $B$ as $\{(a,b)|a \in A, b \in B\}$
- How do we extend this to n-tuples?

# Cartesian products: the idea



# How to build on this

- We actually only have half of the foundations (set theory), we will still need *mathematical logic* to make this airtight
- **Relations** A (binary) relation $R$ from $X$ to $Y$ is a subset of $X \times Y$.
- The domain is often defined as $\forall a \in X \exists b \in Y : (a, b) \in R$, and the codomain (or range) similarly
- The most important relations are **equality** $=$; **similarity** $\sim$; and **ordering** $>$ or $\geq$
- Within set theory, we have only one primitive relation, $\in$ (element of), $=$ is defined by ZFC1

# Relations

- Language offers many examples *x causes y; x has y; x is y . . .*
- In fact logical syntax often treats all verbs as a relation between subject and object: *x loves y* is written as xLy or L(x,y)
- More complex verbs may require ternary relations (*Give(x,y,z); Rent(x,y,z,t,p)*)
- Key idea: an ordered triple can be treated with *left association* as ((a,b),c) or with *right association* as (a,(b,c))
- In Kuratowski encoding:
  $\{\{\{a\}\{ab\}\}\{\{\{a\}\{ab\}\}\{\{a\}\{ab\}\}\{c\}\}\}$ or
  $\{\{a\}\{\{a\}\{\{b\}\{bc\}\}\}\}$
- HW3.1 State the definition of what ot means for the style of encodings to be isomorpic, and prove that they are in fact isomorphic

# Major properties of binary relations

- For any relation $R \subset (X \times Y)$ we can produce the *reversal* $R^T$ and the *complement* $R^C$ of $R$
- These are not the same! For equality $=^T$ is $=$ but $=^C$ is $\neq$
- HW3.2 Write formulas for $R^T$ and $R^C$!
- A relation is **reflexive** iff $\forall a \in X : aRa$
- A relation is **irreflexive** iff $\nexists a \in X : aRa$
- A relation is **symmetric** iff $\forall ab : aRb \Rightarrow bRa$
- A relation is **antisymmetric** iff $\forall ab : aRb \wedge bRa \Rightarrow a = b$
- A relation is **transitive** iff $\forall abc : aRb \wedge bRc \Rightarrow aRc$
- Relations are sets: we can do union, intersection, complementation (relative to universe $X \times Y$)
- Composition of relations can also be defined provided types match: $R \subset X \times Y, S \subset Y \times Z$. We say $a(S \circ R)c \Leftrightarrow \exists b : aRb \wedge bSc$

# Major types of binary relations

1. Relations that enjoy the reflexive, symmetrical, and transitive properties are called **equivalence relations**
2. Relations that are reflexive, antisymmetric, and transitive are called **ordering relations**
3. Equivalence relations are covered in Chapter 9 of CPZ
4. They are closely related to partitions: every e.r. corresponds to a partition
5. We discuss CPZ 9.1-3 in class
6. HW3.3 is CPZ9.4
7. Divisibility (CPZ9.6). $a|b \Leftrightarrow b/a \in \mathbb{N}$
8. We do this now for natural numbers $\{1, 2, \ldots\}$, but it extends smoothly to integers $\mathbb{Z}$
9. HW3.4– is CPZ9.8,10,12,14,16,18,20,22