Foundations of Mathematics, Lecture 10

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Truth

- There are two kinds of truth, syntactic and semantic
- We have $\vdash$ 'yields' or 'derives' where $A \vdash B$ means B can be formally derived (proved) from A. For example, in most systems of logic $x = 3 \wedge y = x \vdash y = 3$, but we need a lot of machinery (called *proof theory*) to make this stick. This is pure syntax manipulation: you take formulas and produce new ones by mechanical operations
- We also have $\models$ 'models' where $A \models B$ means that in any model where A is true B is also true. This is more meaningful, but requires *model theory* which spells out the relation between a theory (bunch of formulas) and a set with lots of structure that the formulas are about
- In well-crafted systems $A \vdash B$ implies $A \models B$

The converse is not true!

- In many well-crafted systems (e.g. the first order formulation of Peano Arithmetic) there are statements which are semantically true e.g. PA $\models$ Goodstein's Theorem, but *has no proof there*
- If it has no proof, how do we know it's true? Because in a stronger system (in this case, 2nd order arithmetic) we can prove it
- That the converse is not true for systems endowed with a bit of arithmetic is the celebrated Gödel Incompleteness Theorem
- Our interest here is with the less celebrated, but just as important, Gödel Completeness Theorem
- This says that every formula that is true in all structures is provable
- Wait, how can these both be true? The answer is that PA has more models in first-order axiomatization than in second-order

Propositional logic

- Any statement that can be true or false (in either of the senses discussed above) is called a **proposition**. These come in two basic varieties: *a has property P* and *the relation R holds between some elements*. Examples of the first: *57 is prime*, of the second: $2 + 3 = 4$
- Things that are *not* propositions include imperatives *Go home!* and questions *Where is Johnny?*
- Declarative statements using variables are called **open propositions** 'x is prime'. These can get a truth value either by substitution '17 is prime' and '18 is prime' both have a truth value or by quantification (part of FOL, but not PL)
- ZFC Axiom 3 (comprehension) creates the connection between PL and set theory: for any open sentence $\phi(x, w_1, \dots, w_n)$ and any set A there exists a set B containing all and only those elements x of A for which $\phi(x, w_1, \dots, w_n)$ holds. 'elements of a set satisfying some proposition can be collected in a set'

Boolean operations

- Well, what are operations? Operations are like addition, multiplication, negation...How can we define operations?
- We don't need new machinery! *Binary* operations are **functions** with two variables. *Unary* operations are functions with one variable (minus, reciprocal, ...) *Nullary operations* are functions that don't depend on any variable, **constants**.
- A **structure** is a set S and some operations. For example groups have a nullary operation (the unit e), a unary operation ($^{-1}$), and a binary operation (multiplication) which satisfy some identities (group axioms). On occasion, we don't insist that an operation be everywhere defined.
- One set of operations that matters in PL are the Boolean $\neg, \wedge, \vee$
- These are 'truth functional' – only the truth of the operands matters for establishing the truth of the result

Implication in PL

1. We define $P \to Q$ by $\neg P \vee Q$
2. This has interesting consequences, the most import being *ex falso quidlibet* 'everything follows from a false statement' or *if you start with a false premiss, you can derive any conclusion from it*
3. We also define truth-functional equivalence $p \equiv Q$ by $P \to Q \wedge Q \to P$
4. Remember sets satisfied the de Morgan identities $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$? In PL they are satisfied by $\neg(A \vee B) = \neg A \wedge \neg B$ and $\neg(A \wedge B) = \neg A \vee \neg B$

Proof theory

1. The goal: mechanical checking of proofs
2. We will manipulate proofs by elementary steps
3. Just one example: Gentzen 'natural deduction' and 'sequent calculus'
4. Every step fits the form $A_1, \ldots, A_n \vdash B_1, \ldots, B_k$ The sides are called *sequents*. WARNING: Left side interpreted conjunctively (',' means $\wedge$), right side interpreted disjunctively (',' means $\vee$) Natural deduction is the special case when $k = 1$
5. Why is this a good trick? Because implication $A \to B$ means $\neg A \vee B$ so when we move stuff to the other side it changes "sign" as in arithmetic. Just one axiom $p, r \vdash q, r$
6. Manipulation rules are written above and below a horizantal line

   e.g. $L \wedge$ rule: $$\frac{\Gamma, A \wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$$

Left:	Right:
$L\wedge$ rule: $\dfrac{\Gamma, A \wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$	$R\wedge$ rule: $\dfrac{\Gamma \vdash \Delta, A \wedge B}{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}$
$L\vee$ rule: $\dfrac{\Gamma, A \vee B \vdash \Delta}{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}$	$R\vee$ rule: $\dfrac{\Gamma \vdash \Delta, A \vee B}{\Gamma \vdash \Delta, A, B}$
$L\rightarrow$ rule: $\dfrac{\Gamma, A \rightarrow B \vdash \Delta}{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}$	$R\rightarrow$ rule: $\dfrac{\Gamma \vdash \Delta, A \rightarrow B}{\Gamma, A \vdash \Delta, B}$
$L\neg$rule: $\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma \vdash \Delta, A}$	$R\neg$rule: $\dfrac{\Gamma \vdash \Delta, \neg A}{\Gamma, A \vdash \Delta}$

Model theory

- We build *model structures* as sets endowed with the right kind of relations
- For example, a model of the natural numbers is a set M endowed with a succession relation $s \subset M \times M$ and satisfying the Peano axioms
- An *interpretation* is a mapping from formulas to a model
- Key notion: *elementary equivalence* holds between structures if they satisfy the same FOL sentences
- Lowenheim-Skolem theorem: if a theory has an infinite model, it has a countably infinite model 'downward L-S'
- If it has a countable model, it has a model at any cardinality 'upward L-S'

Homework

- HW10.1-5: CPZ 2.8; 2.26; 2.32; 2.44; 2.59
- HW10.6: Learn the `tabular` latex environment and build a truth table with the following columns:
  $P, Q, \neg P, \neg Q, P \rightarrow Q, \neg Q \rightarrow \neg P$
- HW10.7-11: CPZ 11.3; 11.5; 11.13; 11.21; and 11.31