Foundations of Mathematics, Lecture 9

András Kornai

BMETE91AM35 Fall 2022-23

Plan of this class

- First half of class: Midterm discussion
- Second half of class: cardinality (CPZ Ch 11)
- Homework: CPZ 4.7; 4.10; 4.13; 4.22; 4.39; 4.51; 4.96; 4.104

The size of sets

- In the finite case: just count the elements. Empty set has zero elements. One-member sets (or their one member) are often called *singleton*s
- The set of natural numbers $\mathbb{N}$ has size $\aleph_0$. This is the first (smallest) infinite cardinality. Sets of this size are called *denumerably infinite*
- Two sets are of the same size (same cardinality) iff there is a bijection between them
- Natural numbers $\mathbb{N}$ and integers $\mathbb{Z}$ have the same cardinality
- And so does the set of rationals $\mathbb{Q}$
- $\mathbb{R}$ is bigger, it's cardinality is conventionally denoted $\mathfrak{c}$ and called *continuum*
- Any real interval, be it limited on one side or both or none, has the same cardinality $\mathfrak{c}$

Existence proof by cardinality argument

1. Important theorem: $\mathfrak{c} > \aleph_0$ Proof: by diagonalization
2. If a set X has cardinality $> \aleph_0$, and a property P is enjoyed only by denumerably many members of X, it follows that *there must be* elements h of X for which $P(h)$ is false
3. Example: let X be $\mathbb{R}$, and P be $\exists p, q \in \mathbb{Z} : x = p/q$ (i.e. $P(h)$ means 'h is rational').
4. Using the above we can *prove* the existence of irrational numbers, but we cannot *construct* one
5. We know how to construct irrationals e.g. $\sqrt{2}$ The classic proof of $\sqrt{2} \neq p/q$ is by "minimum counterexamle" see CPZ 6.4
6. But often we have pure existence proofs that offer no construction
7. Is Bolzano's theorem constructive?

Logic

- To define a logic we will need four things:
- A language to write formulas
- A notion of truth
- A notion of what the formulas mean ‘model theory’
- A deduction procedure ‘proof theory’
- We will discuss three main varieties: propositional, first order, and higher order logic
- We begin at the middle, even though both propositional and higher-order systems are substantially simpler