Foundations of Mathematics, Lecture 6

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Plan of this class

- Administrative things, homework stuff. Midterm two weeks from now, October 25 Within the next 24 hours (Oct 26) you can submit LaTeXversion of your solutions for extra points (but you can get 100% without this step)
- Operations, structures
- With lecture 7 we'll be moving to the 2nd pillar, logic

Operations

- We know many kinds: operations on numbers, operations on sets, operations on functions, on logical formulas, etc etc
- The most typical case is *binary*: take two inputs X and Y, and produce a single output $X + Y$, $X \cup Y$, $X \circ Y$, $X \vee Y$, ...
- Binary operations are functions with two variables
- *Unary* operations are like negative $-X$, reciprocal X^{-1}, negation, etc. These are functions with one variable i.e. binary relations i.e subsets of S^2
- *Nullary* operations are functions with no variable, i.e. *constants*, one-member subsets of S^1. Why one-member (singleton)? Because the output is unique.
- Typical examples are "distinguished elements" such as arithmetic 0 or 1, the truth values, etc.
- There can be higher *arity* operations, e.g. tern*ary* (function with 3 variables) etc. There can also be operations involving elements from more than one structure

Structures

- A structure is a set called the 'base set' endowed with some operations (typically only finitely many) each with a given arity.
- These must satisfy the axioms that characterize the structure
- Well-known examples: groups, rings, fields, Boolean algebras
- We start with *semigroups* which have one binary operation satisfying $(ab)c = a(bc)$ (associativity)
- If we add one nullary operation e that satisfies $ea = ae = a$ we have a *monoid*
- If we also add a unary operation $^{-1}$ that satisfied $aa^{-1} = e$ we have a *group*

Group theory: this is where the fun begins

- The *order* of a group is the cardinality of its base set $|G|$
- The *order* of an element a is the smallest n for which $a^n = e$
- Theorem (Cauchy) in a finite group the order of any element divides the order of the group HW: try to prove this on your own
- Generators