

# FOUNDATIONS OF MATHEMATICS, LECTURE 5

András Kornai

BMETE91AM35 Fall 2022-23

# ADMINISTRATIVE MATTERS

- Missing students (in Neptun, but no homeworks/attendance)  
Khuwayrah Omar Rafat Kamel GACKEE  
Kulakov Anatolii OX20NW
- Some people missing one homework, some are missing two!  
*Please do them*
- Homeworks due Saturday COB (6pm). Late homeworks will be worth less
- Please keep format: email with Subject: FOM, file attached is HWnn\_NEPTUN.pdf
- All material available at  
<https://kornai.com/2022/FoundationsOfMathematics>

# FUNCTIONS AS RELATIONS

- All functions are relations, but not all relations are functions!
- The big difference is that functions have *unique output*, a relation  $F$  will be called a function only if  $aFb \wedge aFc \Rightarrow b = c$
- Definitions of domain, codomain, range, and composition are the same. We don't write  $30^\circ \cos = \sqrt{3}/2$ , we write  $\cos 30^\circ = \sqrt{3}/2$
- CPZ devotes Chapter 10 to functions, we will cover this in class today, but the entire chapter is **homework to read**. **Exercises similar to those in CPZ Ch 1, 9, and 10 will be on the midterm**
- Composition of functions is just like composition of relations: if  $f : A \rightarrow B$  and  $g : B \rightarrow C$  then  $g \circ f : A \rightarrow C$
- Sometimes (often) more lax terminology is used, permitting functions to be defined only on a subset of their domain. For example, most people will talk about  $\sqrt{\phantom{x}}$  as an  $\mathbb{R} \rightarrow \mathbb{R}$  function, even though its *natural domain* is only  $\mathbb{R}_0^+$
- Other tricky point about  $\sqrt{\phantom{x}}$  is that output (depending on definition) is not unique

# MAIN FUNCTION TYPES

- Functions are *defined* or *given* by their graphs, which are the set of (input, output) pairs. But we often think of functions as little machines that take some input and produce some output
- The input may be of different type than the output. Examples: distance travelled as a function of time; temperature as a function of space; force of gravity as a function of masses and distance, ...
- **Multivariate** functions don't depend on a single variable but several. For example, current is a function of both voltage and resistance (Ohm's Law)
- **Vector-valued** sometimes functions produce a  $k$ -tuple of values simultaneously. For example, at any given point in space gravity has both a magnitude and a direction (total of four numbers)
- These can happen at the same time: functions from  $n$ -tuples to  $k$ -tuples are often used

# VARIETY OF FUNCTIONS

- The central types are **numerical functions** from numbers to numbers. You will be seeing a lot of examples of *arithmetic* functions: domain  $\mathbb{N}$  but range can be  $\mathbb{R}$  or even  $\mathbb{C}$
- Also very frequent are **real functions** with domain and range  $\mathbb{R}$
- You will love **complex functions** with domain and range  $\mathbb{C}$
- **Functionals** are functions whose domain are functions, and range is typically  $\mathbb{R}$  or  $\mathbb{C}$
- **Operators** are functions from functions to functions
- All of these are heavily used in physics/engineering
- But there is more! Not all functions involve numbers, for example the truth function maps formulas onto the set  $\{\text{true}, \text{false}\}$
- We will also have a lot to say about **operations** in algebra

# MAIN PROPERTIES OF FUNCTIONS

- 1 Injective: different  $x$ -es map on different  $y$ -s:  
 $f(x) = f(y) \Rightarrow x = y$
- 2 Surjective: codomain = range (codomain  $\supset$  range is true by definition)
- 3 Bijective: both injective and surjective
- 4 **Theorem:** a function  $f$  is *invertible*  $\Leftrightarrow$  it is *bijective*
- 5 **Proof:** We need to prove both  $\Rightarrow$  and  $\Leftarrow$ . For  $\Rightarrow$  we need to *verify* that the bijective properties follow from invertibility. For  $\Leftarrow$  we will construct the inverse of a bijective function.
- 6 ( $\Rightarrow$ ) What do we suppose? What do we need to prove?
- 7 ( $\Leftarrow$ ) What do we suppose? What do we need to prove?
- 8 Discussion of CPZ Ex 10.18
- 9 Homework [CPZ 9.78-9.80](#); [10.1-10.9](#)