

FOUNDATIONS OF MATHEMATICS, LECTURE 4

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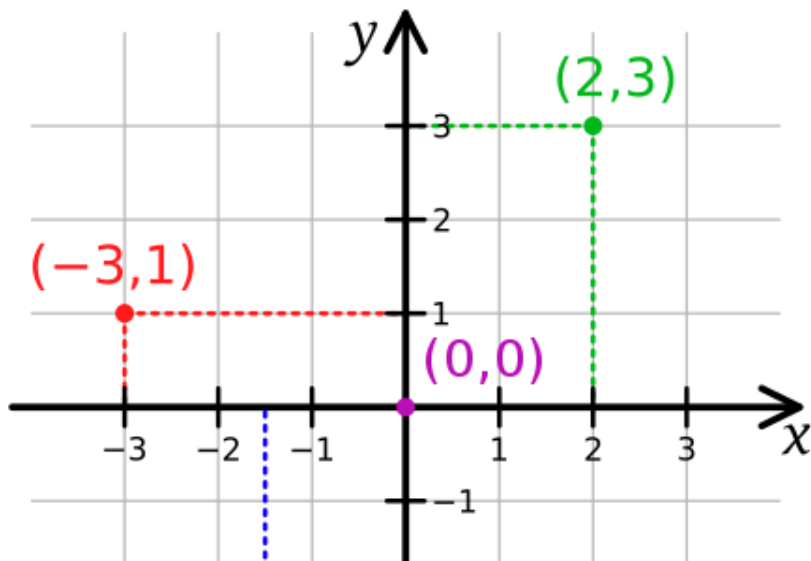
BMETE91AM35 Fall 2022-23

ADMINISTRATIVE MATTERS

- Missing students (in Neptun, but no homeworks/attendance)

Run Bunthorng	BKWI51
Munkhsaikhan Bayarjargal	FPQV85
Khuwayrah Omar Rafat Kamel	GACKEE
Symonenko Mykyta	H8IUDR
Kulakov Anatolii	OX20NW
- Some people missing one homework *please do them!*
- All material available at
<https://kornai.com/2022/FoundationsOfMathematics>

CARTESIAN PRODUCTS: THE IDEA



CARTESIAN PRODUCTS: IMPLEMENTING THE IDEA

- We need *ordered pairs*, where $(1, 2) \neq (2, 1)$
- One approach (Hausdorff 1914): number the coordinates: define (x, y) by $\{\{x, 1\}, \{y, 2\}\}$. This gets trickier when x, y are numbers, and at any rate it presupposes the idea of a number
- The standard approach (Kuratowski 1921)
 $(x, y) \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$
- Variants are possible, don't bring much novelty
- Now we can build the cartesian or *direct* product $A \times B$ of two sets A and B as $\{(a, b) | a \in A, b \in B\}$
- How do we extend this to n-tuples?
- This will lead us to a bunch of useful definitions, for *relation*, *function*, *operation* etc.

WE HAVE THE FOUNDATIONS, LET'S BUILD THE HOUSE!

- We actually only have half of the foundations (set theory), we will still need *mathematical logic* to make this airtight, but we can start building
- **Relations** A (binary) relation R with domain X and codomain Y is a subset of $X \times Y$ such that $\forall a \in X \exists b \in Y : (a, b) \in R$, and conversely, any such subset is a relation with domain X and codomain Y
- The most important relations are **equality** $=$; **similarity** \sim ; and **ordering** $>$ or \geq
- Within set theory, we have only two relations treated as primitive: \in (element of) and $=$ (equal)
- These two are linked by ZFC1
- Difference between codomain and range!

RELATIONS

- Language offers many examples x causes y ; x has y ; x is y . . .
- In fact logical syntax often treats all verbs as a relation between subject and object: x loves y is written as xLy or $L(x,y)$
- More complex verbs may require ternary relations ($Give(x,y,z)$; $Rent(x,y,z,t,p)$)
- Key idea: an ordered triple can be treated with *left association* as $((a,b),c)$ or with *right association* as $(a,(b,c))$
- In Kuratowski encoding:
 $\{\{\{a\}\{ab\}\}\{\{\{a\}\{ab\}\}\{\{a\}\{ab\}\}\{c\}\}\}$ or
 $\{\{a\}\{\{a\}\{\{b\}\{bc\}\}\}\}$
- HW4.1 State the definition of what it means for the style of encodings to be isomorphic, and prove that they are in fact isomorphic

MAJOR PROPERTIES OF BINARY RELATIONS

- For any relation $R \subset (X \times Y)$ we can produce the *reversal* R^T and the *complement* R^C of R
- These are not the same! For equality $=^T$ is $=$ but $=^C$ is \neq
- HW4.2 Write formulas for R^T and R^C !
- A relation is **reflexive** iff $\forall a \in X : aRa$
- A relation is **irreflexive** iff $\nexists a \in X : aRa$
- A relation is **symmetric** iff $\forall ab : aRb \Rightarrow bRa$
- A relation is **antisymmetric** iff $\forall ab : aRb \wedge bRa \Rightarrow a = b$
- A relation is **transitive** iff $\forall abc : aRb \wedge bRc \Rightarrow aRc$
- Relations are sets: we can do union, intersection, complementation (relative to universe $X \times Y$)
- Composition of relations can also be defined provided types match: $R \subset X \times Y, S \subset Y \times Z$. We say $a(S \circ R)c \Leftrightarrow \exists b : aRb \wedge bSc$

MAJOR TYPES OF BINARY RELATIONS

- 1 Relations that enjoy the reflexive, symmetrical, and transitive properties are called **equivalence relations**
- 2 Relations that are reflexive, antisymmetric, and transitive are called **ordering relations**
- 3 Equivalence relations are covered in Chapter 9 of CPZ
- 4 They are closely related to partitions: every e.r. corresponds to a partition
- 5 We discuss CPZ 9.1-3 in class
- 6 HW4.3 is CPZ9.4
- 7 Divisibility (CPZ9.6). $a|b \Leftrightarrow b/a \in \mathbb{N}$
- 8 We do this now for natural numbers $\{1, 2, \dots\}$, but it extends smoothly to integers \mathbb{Z}
- 9 HW4.4– is CPZ9.8,10,12,14,16,18,20,22