Foundations of Mathematics, Lecture 2

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About the 0th test

- Nobody got all problems right, some got all wrong, most people in between
- Problem 1 was the easiest, BUT lots of people gave no justification!
- Problem 2 solved by exactly one person
- We will start building LaTeXskills: people will have to typeset ZFC
- We will start using Chartrand-Polimeni-Zhang

Zermelo-Fraenkel Axioms

1. Axiom of Extensionality: If X and Y have the same elements, then $X = Y$.

$$\forall u\,(u \in X \equiv u \in Y) \Rightarrow X = Y. \tag{1}$$

2. Axiom of the Unordered Pair: For any a and b there exists a set $\{a,\ b\}$ that contains exactly a and b. (also called Axiom of Pairing)

$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \equiv (x = a \vee x = b)). \tag{2}$$

3. Axiom of Subsets: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \varphi(u,\ p)\}$ that contains all those $u \in X$ that have the property φ. (also called Axiom of Separation or Axiom of Comprehension)

$$\forall X\,\forall p\,\exists Y\,\forall u\,(u \in Y \equiv (u \in X \wedge \varphi(u,\ p))). \tag{3}$$

4. Axiom of the Sum Set: For any X there exists a set $Y = \bigcup X$, the union of all elements of X. (also called Axiom of Union)

$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv \exists z\,(z \in X \wedge u \in z)). \tag{4}$$

5. Axiom of the Power Set: For any X there exists a set $Y = P(X)$, the set of all subsets of X.

$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv u \subseteq X). \tag{5}$$

6. Axiom of Infinity: There exists an infinite set.

$$\exists S\,[\emptyset \in S \wedge (\forall x \in S)\,[x \bigcup \{x\} \in S]]. \tag{6}$$

7. Axiom of Replacement: If F is a function, then for any X there exists a set $Y = F[X] = \{F(x) : x \in X\}$.

$$\begin{aligned}&\forall x\,\forall y\,\forall z\,[\varphi(x,\ y,\ p) \wedge \varphi(x,\ z,\ p) \Rightarrow y = z]\\ &\Rightarrow \forall X\,\exists Y\,\forall y\,[y \in Y \equiv (\exists x \in X)\,\varphi(x,\ y,\ p)].\end{aligned} \tag{7}$$

8. Axiom of Foundation: Every nonempty set has an $\in$-minimal element. (also called Axiom of Regularity)

$$\forall S\,[S \neq \emptyset \Rightarrow (\exists x \in S)\,S \bigcap x = \emptyset]. \tag{8}$$

9. Axiom of Choice: Every family of nonempty sets has a choice function.

$$\forall x \in a\,\exists A(x,\ y) \Rightarrow \exists y\,\forall x \in a\,A(x,\ y(x)). \tag{9}$$

Latex basics

- You need to install latex on your laptop. Visit https://www.latex-project.org/get/ and choose one (TexLive for Linux, MacTex for Mac, or MikTex for Windows), download, and try it out!
- Some of these come with a full graphical interface, but you can use just an ordinary plain text editor (emacs, vi, vim, nano, etc) in a terminal window
- There is a write-compile-debug cycle: start with file.tex, produce file.pdf, test, repeat (a bit more complex if a bibliography is involved)
- Your first homework will be to write ZFC in LaTeX
- Yes, you can do this in your own language, not just English!
- Download `zfcskel.tex` from the course webpage and edit that. Start with English even if you want to learn LaTeXfor some other script or language

The basic structure of your document

```
\documentclass{article}
\usepackage{colortbl}
\author{YOUR NAME}
\title{The ZFC axioms of set theory}
\date{}
\begin{document}
\maketitle

\begin{enumerate}

\item {\color{green} Extensionality} If $X$ and $Y$ ...
\begin{equation}
\forall u ...
\end{equation}

\item {\color{green} Unordered Pair} For any $a$ and $b$
\begin{equation}
\forall a \forall b
\end{equation}
...

\item {\color{green} Choice}  Every family of nonempty sets...
\begin{equation}
\forall x \in a \exists A(x,y) \Rightarrow ...
\end{equation}
\end{enumerate}

\end{document}
```

Reasons why all mathematicians use LaTeX

1. Typesetting formulas/equations is *hard*, LaTeXmakes is easy
2. Too many symbols are used to fit on any keyboard. It is much better to learn a few names, like `\forall` for $\forall$, `\exists` for $\exists$, ...
3. Clear separation of format and content
4. Clear separation of "object language" and "metalanguage"
5. Can do all kinds of stuff, but runs everywhere
6. Can do much better typesetting than Word or really anything else
7. Well integrated with rest of the ecosystem (e.g. MathJax for webpages)

The first problem from the 0th test

Three subsets A, B, and C of $\{1,2,3,4,5\}$ have the same cardinality. Furthermore

- A 1 belongs to A and B but not to C
- B 2 belongs to A and C but not to B
- C 3 belongs to A and exactly one of B and C
- D 4 belongs to an even number of A, B, and C
- E 5 belongs to an odd number of A, B, and C
- F The sums of the elements in two of the sets A, B, and C differ by 1

	1	2	3	4	5
A	$+$	$+$	$+$	$0/2$	$1/3$
B	$+$	$-$	$\vee$	$0/2$	$1/3$
C	$-$	$+$	$\wedge$	$0/2$	$1/3$

Applying the other conditions

- So far, we know that A has at least 3 elements. Can it have 5?
- No, because in that case B and C must also have 5 elements, so they all would be the same set by Axiom 1.
- Can they all have four? No, because that would require 4 and 5 to appear in both B and C, and that would still not be enought, because 3 appears in only one of these!
- So we are done with A, and we now know that 5 cannot appear in A, so condition E means we must have 1 occurrence of 5 (either in B or in C, not both)

	1	2	3	4	5
A	$+$	$+$	$+$	$-$	$-$
B	$+$	$-$	$\vee$	$0/2$	$\wedge$
C	$-$	$+$	$\wedge$	$0/2$	$\vee$

Applying the other conditions

Since B will have one of 3 and 5, and C will have the other, we must have 4 both in B and C. This leaves us two possibilities:

	1	2	3	4	5
A	+	+	+	−	−
B	+	−	+	+	−
C	−	+	−	+	+

	1	2	3	4	5
A	+	+	+	−	−
B	+	−	−	+	+
C	−	+	+	+	−

So the sums of the numbers in the sets will be

	1	2	3	4	5	
A	+	+	+	−	−	6
B	+	−	+	+	−	8
C	−	+	−	+	+	10

	1	2	3	4	5	
A	+	+	+	−	−	6
B	+	−	−	+	+	10
C	−	+	+	+	−	9

Homework

- Produce a .tex and .pdf version of the ZFC axioms given on Slide 2 above. You can start with `zfcskel.tex` from the course webpage