Foundations of Mathematics, Lecture 11

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Plan of this class

- Solutions to problems
- Logic cont'd

Logic

- To define a logic we will need four things:
- A language to write formulas
- A notion of truth
- A notion of what the formulas mean ‘model theory’
- A deduction procedure ‘proof theory’
- We will discuss three main varieties: propositional, first order, and higher order logic

Truth

- There are two kinds of truth, syntactic and semantic
- We have $\vdash$ 'yields' or 'derives' where $A \vdash B$ means B can be formally derived (proved) from A. For example, in most systems of logic $x = 3 \wedge y = x \vdash y = 3$, but we need a lot of machinery (called *proof theory*) to make this stick. This is pure syntax manipulation: you take formulas and produce new ones by mechanical operations
- We also have $\models$ 'models' where $A \models B$ means that in any model where A is true B is also true. This is more meaningful, but requires *model theory* which spells out the relation between a theory (bunch of formulas) and a set with lots of structure that the formulas are about
- In well-crafted systems $A \vdash B$ implies $A \models B$

The converse is not true!

- In many well-crafted systems (e.g. the first order formulation of Peano Arithmetic) there are statements which are semantically true e.g. PA $\models$ Goodstein's Theorem, but *has no proof there*
- If it has no proof, how do we know it's true? Because in a stronger system (in this case, 2nd order arithmetic) we can prove it
- That the converse is not true for systems endowed with a bit of arithmetic is the celebrated Gödel Incompleteness Theorem
- Our interest here is with the less celebrated, but just as important, Gödel Completeness Theorem
- This says that every formula that is true in all structures is provable
- Wait, how can these both be true? The answer is that PA has more models in first-order axiomatization than in second-order

Propositional logic

- Any statement that can be true or false (in either of the senses discussed above) is called a **proposition**. These come in two basic varieties: *a has property P* and *the relation R holds between some elements*. Examples of the first: *57 is prime*, of the second: $2 + 3 = 4$
- Things that are *not* propositions include imperatives *Go home!* and questions *Where is Johnny?*
- Declarative statements using variables are called **open propositions** 'x is prime'. These can get a truth value either by substitution '17 is prime' and '18 is prime' both have a truth value or by quantification (part of FOL, but not PL)
- ZFC Axiom 3 (comprehension) creates the connection between PL and set theory: for any open sentence $\phi(x, w_1, \ldots, w_n)$ and any set A there exists a set B containing all and only those elements x of A for which $\phi(x, w_1, \ldots, w_n)$ holds. 'elements of a set satisfying some proposition can be collected in a set'

Boolean operations

- Well, what are operations? Operations are like addition, multiplication, negation... How can we define operations?
- We don't need new machinery! *Binary* operations are **functions** with two variables. *Unary* operations are functions with one variable (minus, reciprocal, ...) *Nullary operations* are functions that don't depend on any variable, **constants**.
- A **structure** is a set S and some operations. For example groups have a nullary operation (the unit e), a unary operation ($^{-1}$), and a binary operation (multiplication) which satisfy some identities (group axioms). On occasion, we don't insist that an operation be everywhere defined.
- One set of operations that matters in PL are the Boolean $\neg, \wedge, \vee$
- These are 'truth functional' – only the truth of the operands matters for establishing the truth of the result

Implication in PL

1. We define $P \to Q$ by $\neg P \vee Q$
2. This has interesting consequences, the most import being *ex falso quidlibet* 'everything follows from a false statement' or *if you start with a false premiss, you can derive any conclusion from it*
3. We also define truth-functional equivalence $p \equiv Q$ by $P \to Q \wedge Q \to P$
4. Remember sets satisfied the de Morgan identities $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$? In PL they are satisfied by $\neg(A \vee B) = \neg A \wedge \neg B$ and $\neg(A \wedge B) = \neg A \vee \neg B$

Homework

- Homework for next week: CPZ 2.8; 2.26; 2.32; 2.44; 2.59 and HW11.6: Learn the `tabular` latex environment and build a truth table with the following columns: $P, Q, \neg P, \neg Q, P \to Q, \neg Q \to \neg P$
- (Only pdf files (latex source not required), **Subject: FOM NEPTUN HW11**