Foundations of Mathematics, Lecture 10

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Plan of this class

- Finish with cardinality (CPZ ch 11)

The size of sets

- Finite, denumerably infinite, continuum, etc
- We say $|A| = |B|$ if there is a bijective mapping $i : A \to B$
- We say $|A| \leq |B|$ if there is an injective mapping $i : A \to B$
- Having the same cardinality is an equivalence relation
- Having smaller-or-equal cardinality is an ordering relation
- The missing piece: if $|A| \leq |B| \wedge |B| \leq |A|$ we have $|A| = |B|$ This is called the (Cantor-)Schröder-Bernstein Theorem HW10.1 – try to prove this without looking at the proof in CPZ. If you can't finish the proof, write it down where you failed.
- There is no largest cardinality! $|2^P| > |P|$

Existence proof by cardinality argument

1. Important theorem: $\mathfrak{c} > \aleph_0$ Proof: by diagonalization
2. If a set X has cardinality $> \aleph_0$, and a property P is enjoyed only by denumerably many members of X, it follows that *there must be* elements h of X for which $P(h)$ is false
3. Example: let X be $\mathbb{R}$, and P be $\exists p, q \in \mathbb{Z} : x = p/q$ (i.e. $P(h)$ means 'h is rational').
4. Using the above we can *prove* the existence of irrational numbers, but we cannot *construct* one
5. We know how to construct irrationals e.g. $\sqrt{2}$ The classic proof of $\sqrt{2} \neq p/q$ is by "minimum counterexamle" see CPZ 6.4
6. But often we have pure existence proofs that offer no construction
7. HW 10.2 Prove Bolzano's theorem. Is your proof constructive?
8. HW 10.3 Find $p, q \in \mathbb{Z}$ such that $0.514141414... = p/q$