Solutions to 1st Midterm

1. (a) $\emptyset, \{\emptyset\}$
   (b) $|A| = 3$
   (c) $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$
   (d) $\emptyset$
   (e) $\{\emptyset\}$
   (f) $\emptyset, \{\emptyset\}$
   (g) A
   (h) A
   (i) A

2. (a) No.
   (b) Yes.
   (c) Yes.

3. (a) Suppose f and g are injective. $g(f(x)) = g(f(y))$, and since g is injective, $f(x) = f(y)$. f is also injective, so $x = y$, thus $g \circ f$ is injective. Suppose f and g are surjective. $\forall z \in C : \exists y \in B : g(y) = z$. Since f is surjective, $\forall y \in B : \exists x \in A : f(x) = y$. Therefore $\forall z \in C : \exists x : g(f(x)) = z$, thus $g \circ f$ is surjective. $g \circ f$ is both injective and surjective, thus bijective. This statement is **true**.
   (b) Let $A = B = C = \mathbb{N}; f(x) = 1, g(x) = x$. g is surjective, since $\text{ran}(g) = \mathbb{N}$, but $\text{ran}(g(f(x))) = 1$. This statement is **false**.
   (c) Let $A = B = C = \mathbb{N}; f(x) = 1, g(x) = x$. g is injective, since $g(x) = g(y) \Rightarrow x = y$, but $f(g(x)) = f(g(y)) \not\Rightarrow x = y$ for $x = 1$ and $y = 2$. This statement is **false**.
   (d) Let $A = \mathbb{N}, B = \{0, 1, 2, \ldots 10\}, C = \{1\}; f(x) = x \bmod 10, g(x) = 1$. f is not surjective, since $\text{ran}(f) = \{0 \ldots 9\} \neq B$, but $\text{ran}(g(f(x))) = \{1\} = C$. This statement is **true**.
   (e) Let A, B, C be arbitrary sets, f is not injective, so $\exists x, y \in A : f(x) = f(y) = a \wedge x \neq y$. In this case, $g \circ f : A \to C$. $f(x) = f(y)$, thus $(g \circ f)(x) = (g \circ f)(y)$, but $x \neq y$. $g \circ f$ is not injective, this statement is **false**.

4. [1] = {1,4,5}, [2] = {2,6}, [3] = {3},
   [4] = {1,4,5}, [5] = {1,4,5}, [6] = {2,6}.
   Then we get R = {(1,1), (1,4), (1,5), (2,2), (2,6), (3,3), (4,1), (4,4), (4,5), (5,1), (5,4), (5,5),(6,2), (6,6)}