Foundations of Mathematics, Lecture 9

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Class and homework

- At the end we will spend part of class discussing problems from Chapter 2 of CPZ.
- Homework for next week: CPZ 2.8; 2.26; 2.32; 2.44; 2.59 and the following: HW8.6: Learn the latex `tabular` environment and create truth table with columns for $P, Q, \neg P, \neg Q, P \rightarrow Q, \neg Q \rightarrow \neg P$
- Remember to send pdf (not the latex source) with **Subject: FOM NEPTUN HW9**

Refresher: (in)completeness

- (In)completeness is about the relative power of the $\vdash$ and $\models$ relation. When they have the same power the system is complete, when $\vdash$ is weaker it's called incomplete
- In many well-crafted systems (e.g. the first order formulation of Peano Arithmetic) there are statements which are semantically true e.g. PA $\models$ Goodstein's Theorem, but *has no proof there*
- If it has no proof, how do we know it's true? Because in a stronger system (in this case, 2nd order arithmetic) we can prove it
- Our interest will be with the less celebrated, but just as important, Gödel Completeness Theorem
- This says that every formula that is true in all structures is provable
- But we give an example of incompleteness that is a lot easier to understand than Gödel's original proof, and does not involve Gödel numbering

Goodstein's theorem

- When we write a natural number n in base b this means we express it as $\sum_{i=0}^{k} c_{j_i} b^{e_{j_i}}$ where the digits c_{j_i} are between 1 and $b-1$ and the exponents e_{j_i} are arbitrary nonnegative integers.
- In *hereditary* base b notation we force the exponents to be also written in base b and the exponents therein, and so on, until all digits are $< b$ everywhere.
- The *Goodstein sequence* $G(n)(k)$ for a number n begins with $G(n)(2) = n$, written in hereditary base 2. Next replace each occurence of 2 by 3 (in general, each occurrence of b by $b+1$) and subtract 1. Since you increase the base and substract only one the sequence increases very very fast.
- **Goodstein's Theorem** Every Goodstein sequence terminates in 0 in finitely many steps
- Once you have the theory of *ordinals* at your disposal, the theorem is easy to prove. But no proof without such powerful theory exists (Kirby and Paris 1982).

Propositional logic

- Any statement that can be true or false (in either of the senses discussed above) is called a **proposition**. These come in two basic varieties: *a has property P* and *the relation R holds between some elements*. Examples of the first: *57 is prime*, of the second: $2 + 3 = 4$
- Things that are *not* propositions include imperatives *Go home!* and questions *Where is Johnny?*
- Declarative statements using variables are called **open propositions** 'x is prime'. These can get a truth value either by substitution '17 is prime' and '18 is prime' both have a truth value or by quantification (part of FOL, but not PL)
- ZFC Axiom 3 (comprehension) creates the connection between PL and set theory: for any open sentence $\phi(x, w_1, \ldots, w_n)$ and any set A there exists a set B containing all and only those elements x of A for which $\phi(x, w_1, \ldots, w_n)$ holds. 'elements of a set satisfying some proposition can be collected in a set'

Boolean operations

- Well, what are operations? Operations are like addition, multiplication, negation…How can we define operations?
- We don't need new machinery! *Binary* operations are **functions** with two variables. *Unary* operations are functions with one variable (minus, reciprocal, …) *Nullary operations* are functions that don't depend on any variable, **constants**.
- A **structure** is a set S and some operations. For example groups have a nullary operation (the unit e), a unary operation ($^{-1}$), and a binary operation (multiplication) which satisfy some identities (group axioms). On occasion, we don't insist that an operation be everywhere defined.
- One set of operations that matters in PL are the Boolean $\neg, \wedge, \vee$
- These are 'truth functional' – only the truth of the operands matters for establishing the truth of the result

Implication in PL

1. We define $P \to Q$ by $\neg P \vee Q$
2. This has interesting consequences, the most import being *ex falso quidlibet* ‘everything follows from a false statement’ or *if you start with a false premiss, you can derive any conclusion from it*
3. We also define truth-functional equivalence $P \equiv Q$ by $P \to Q \wedge Q \to P$
4. Remember sets satisfied the de Morgan identities $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$? In PL they are satisfied by $\neg(A \vee B) = \neg A \wedge \neg B$ and $\neg(A \wedge B) = \neg A \vee \neg B$
5. HW8: CPZ Chapter 2, exercises