Foundations of Mathematics, Lecture 8

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Truth

- There are two kinds of truth, syntactic and semantic
- We have $\vdash$ 'yields' or 'derives' where $A \vdash B$ means B can be formally derived (proved) from A. For example, in most systems of logic $x = 3 \wedge y = x \vdash y = 3$, but we need a lot of machinery (called *proof theory*) to make this stick. This is pure syntax manipulation: you take formulas and produce new ones by mechanical operations
- We also have $\models$ 'models' where $A \models B$ means that in any model where A is true B is also true. This is more meaningful, but requires *model theory* which spells out the relation between a theory (bunch of formulas) and a set with lots of structure that the formulas are about
- In well-crafted systems $A \vdash B$ implies $A \models B$

The converse is not true!

- In many well-crafted systems (e.g. the first order formulation of Peano Arithmetic) there are statements which are semantically true e.g. PA $\models$ Goodstein's Theorem, but *has no proof there*
- If it has no proof, how do we know it's true? Because in a stronger system (in this case, 2nd order arithmetic) we can prove it
- That the converse is not true for systems endowed with a bit of arithmetic is the celebrated Gödel Incompleteness Theorem
- Our interest here is with the less celebrated, but just as important, Gödel Completeness Theorem
- This says that every formula that is true in all structures is provable
- Wait, how can these both be true? The answer is that PA has more models in first-order axiomatization than in second-order