Foundations of Mathematics, Lecture 5

András Kornai

BMETE91AM35 Fall 2021-22

Administrative matters

- Missing students (in Neptun, but no homeworks/attendance)
  F20ABR Albari Mohamad
  MIEFJD Lomia Ketevan
  MIEFJD Lorn Sokly
  MIEFJD Nguyen Huy
- Some people missing homework *please do them!* IC3D25 BKCEWI IBZJH9 HCVGL4
- All material available at https://kornai.com/2021/FoundationsOfMathematics

Approx summary of last class

- We actually only have half of the foundations (set theory), we will still need *mathematical logic* to make this airtight, but we can start building
- **Relations** A (binary) relation R with domain X and codomain Y is a subset of $X \times Y$ such that $\forall a \in X \exists b \in Y : (a, b) \in R$, and conversely, any such subset is a relation with domain X and codomain Y
- The most important relations are **equality** $=$; **similarity** $\sim$; and **ordering** $>$ or $\geq$
- Within set theory, we have only two relations treated as primitive: $\in$ (element of) and $=$ (equal)
- These two are linked by ZFC1
- Difference between codomain and range!

Relations

- Language offers many examples *x causes y; x has y; x is y ...*
- In fact logical syntax often treats all verbs as a relation between subject and object: *x loves y* is written as xLy or L(x,y)
- More complex verbs may require ternary relations (*Give(x,y,z); Rent(x,y,z,t,p)*)
- Key idea: an ordered triple can be treated with *left association* as ((a,b),c) or with *right association* as (a,(b,c))
- In Kuratowski encoding: $\{\{\{a\}\{ab\}\}\{\{\{a\}\{ab\}\}\{\{a\}\{ab\}\}\{c\}\}\}$ or $\{\{a\}\{\{a\}\{\{b\}\{bc\}\}\}\}$
- HW4.1 State the definition of what ot means for the style of encodings to be isomorpic, and prove that they are in fact isomorphic

Major properties of binary relations

- For any relation $R \subset (X \times Y)$ we can produce the *reversal* R^T and the *complement* R^C of R
- These are not the same! For equality $=^T$ is $=$ but $=^C$ is $\neq$
- HW4.2 Write formulas for R^T and R^C!
- A relation is **reflexive** iff $\forall a \in X : aRa$
- A relation is **irreflexive** iff $\nexists a \in X : aRa$
- A relation is **symmetric** iff $\forall ab : aRb \Rightarrow bRa$
- A relation is **antisymmetric** iff $\forall ab : aRb \wedge bRa \Rightarrow a = b$
- A relation is **transitive** iff $\forall abc : aRb \wedge bRc \Rightarrow aRc$
- Relations are sets: we can do union, intersection, complementation (relative to universe $X \times Y$)
- Composition of relations can also be defined provided types match: $R \subset X \times Y, S \subset Y \times Z$. We say $a(S \circ R)c \Leftrightarrow \exists b : aRb \wedge bSc$

Major types of binary relations

1. Relations that enjoy the reflexive, symmetrical, and transitive properties are called **equivalence relations**
2. Relations that are reflexive, antisymmetric, and transitive are called **ordering relations**
3. Equivalence relations are covered in Chapter 9 of CPZ
4. They are closely related to partitions: every e.r. corresponds to a partition
5. We discuss CPZ 9.1-3 in class
6. HW4.3 is CPZ9.4
7. Divisibility (CPZ9.6). $a|b \Leftrightarrow b/a \in \mathbb{N}$
8. We do this now for natural numbers $\{1, 2, \ldots\}$, but it extends smoothly to integers $\mathbb{Z}$
9. HW4.4– is CPZ9.8,10,12,14,16,18,20,22

Partitioning

- A *partition* of a set A is a family of sets A_α such that for any $\alpha \neq \beta$ we have $A_\alpha \cap A_\beta = \emptyset$ and $\bigcup_\alpha A_\alpha = A$. By definition, we never consider the empty set a part of any partitition, so in the definition we may write "a family of *nonempty* sets"
- The partition can be finite (e.g. the sets of *even* and *odd* numbers partition the set of integers) or infinite
- Partitions are related to *equivalence relations*.
- There are two *trivial* partitions, when all elements are in the same set, and when all go in their own set

Functions as relations

- All functions are relations, but not all relations are functions!
- The big difference is that functions have *unique output*, a relation F will be called a function only if $aFb \wedge aFc \Rightarrow b = c$
- Definitions of domain, codomain, range, and composition are the same. We don't write 30° cos $\sqrt{3}/2$, we write $\cos 30^\circ = \sqrt{3}/2$
- CPZ devotes Chapter 10 to functions, we will cover this in class today, but the entire chapter is homework to read. Exercises similar to those in CPZ Ch 1, 9, and 10 will be on the midterm
- Composition of functions is just like composition of relations: if $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$
- Sometimes (often) more lax terminology is used, permitting functions to be defined only on a subset of their domain. For example, most people will talk about $\sqrt{\ }$ as an $\mathbb{R} \to \mathbb{R}$ function, even though its *natural domain* is only $\mathbb{R}_0^+$
- Other tricky point about $\sqrt{\ }$ is that output (depending on definition) is not unique

Main function types

- Functions are *defined* or *given* by their graphs, which are the set of (input, output) pairs. But we often think of functions as little machines that take some input and produce some output
- The input may be of different type than the output. Examples: distance travelled as a function of time; temperature as a function of space; force of gravity as a function of masses and distance, ...
- **Multivariate** functions don't depend on a single variable but several. For example, current is a function of both voltage and resistance (Ohm's Law)
- **Vector-valued** sometimes functions produce a k-tuple of values simultaneously. For example, at any given point in space gravity has both a magnitude and a direction (total of four numbers)
- These can happen at the same time: functions from n-tuples to k-tuples are often used

Variety of functions

- The central types are **numerical functions** from numbers to numbers. You will be seeing a lot of examples of *arithmetic* functions: domain $\mathbb{N}$ but range can be $\mathbb{R}$ or even $\mathbb{C}$
- Also very frequent are **real functions** with domain and range $\mathbb{R}$
- You will love **complex functions** with domain and range $\mathbb{C}$
- **Functionals** are functions whose domain are functions, and range is typically $\mathbb{R}$ or $\mathbb{C}$
- **Operators** are functions from functions to functions
- All of these are heavily used in physics/engineering
- But there is more! Not all functions involve numbers, for example the truth function maps formulas onto the set $\{$true, false $\}$
- We will also have a lot to say about **operations** in algebra

Main properties of functions

1. Injective: different x-es map on different y-s:
   $f(x) = f(y) \Rightarrow x = y$
2. Surjective: codomain $=$ range (codomain $\supset$ range is true by definition)
3. Bijective: both injective and surjective
4. **Theorem:** a function f is *invertible* $\Leftrightarrow$ it is *bijective*
5. **Proof:** We need to prove both $\Rightarrow$ and $\Leftarrow$. For $\Rightarrow$ we need to *verify* that the bijective properties follow from invertibility. For $\Leftarrow$ we will construct the inverse of a bijective function.
6. ($\Rightarrow$) What do we suppose? What do we need to prove?
7. ($\Leftarrow$) What do we suppose? What do we need to prove?
8. Discussion of CPZ Ex 10.18