

FOUNDATIONS OF MATHEMATICS, LECTURE 3

András Kornai

BMETE91AM35 Fall 2021-22

ABOUT THE 0TH TEST, 2ND PROBLEM

- The task was to (a) *compute* $\cos(20^\circ) \cos(40^\circ) \cos(80^\circ)$ and (b) *prove* that this is exactly $1/8$.
- Computing it to four-five decimals and obtain 0.12500 is trivial
- Many people used a trick, multiplying by, and also dividing by, $\sin(20^\circ)$ and relying on formulas of multiple angles
- This is fine, as long as we can prove these formulas as lemmas
- We will develop a very different proof, which doesn't rely on the axiomatization of geometry that much

PROOF OUTLINE

- We will prove the well-known addition formulas $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- You may have learned the geometric proof for these in high school: for detailed explanation of these there are videos e.g. on Khan Academy
- These actually require proving lots of lemmas which high-school geometry doesn't really teach rigorously
- We will return to this part of the proof when you have already studied [complex numbers](#), [matrix multiplication](#) and [isomorphism](#)
- Once these pieces of algebra are at hand, the addition formulas will be trivial, but the pieces (including the [Euler](#) and [de Moivre formulas](#)) will be super-useful for many other things as well!
- We will need other pieces of algebra, in particular, you will also learn about [polynomials](#) and [structures](#) before we can complete the proof. But these things are again super-useful on their own

DISCUSSION OF L^AT_EX ISSUES

- Anybody has installation issues? We are here to help
- Anybody had serious problems with the ZFC homework? We are here to help
- This is not a course in L^AT_EX, there will be no more specifically L^AT_EX homeworks
- But if you are still missing the ZFC homework, fix this soonest, or it will affect your grade
- All subsequent homeworks are due in L^AT_EX
- Homeworks will be evaluated on content, not formatting, but correct formatting is expected for the rest of your life. You will be grateful that we forced this on you!
- Also, teachers in other courses will love you for it

SET OPERATIONS

- 1 Union, intersection, but no complementation in general – why not?
- 2 Subscripting, indexing
- 3 Partitioning
- 4 Cartesian (also called direct) product
- 5 Ordered pairs by sets
- 6 Alternatives to the standard definition
- 7 Back to direct products

BOOLEAN OPERATIONS

- When there is a base set (such as the set of integers) everything works!
- We get the de Morgan identities: $\overline{A \cup B} = \bar{A} \cap \bar{B}$ and $\overline{A \cap B} = \bar{A} \cup \bar{B}$
- Complementation is an involution $\overline{\bar{A}} = A$
- \cup and \cap are commutative, associative, idempotent
- Two kinds of distributive laws
- There is a 0 (the empty set) and a 1 (the universe)
- Will be generalized to *lattices* later on

TROUBLE WITH NEGATION

- There is no “set of all sets”
- Why not? Because it would not be well founded (doable with AFA, but not with ZFC)
- Because it would give rise to Russel’s Paradox: by Comprehension we could form the set of all sets that don’t contain themselves!
- Paradox can be avoided by (a) positive comprehension (b) type theory
- Most math is done with (b) – types are built into ZFC
- Strongly related to type checking in programming languages
- You can have the *class* of all sets, and other classes (NGB set theory)

SUBSCRIBING, INDEXING

- The basic idea: instead of A, B, C, \dots, Z and running out of letters after 26, let's do A_1, A_2, \dots, A_{777} because we never run out of numbers
- But what if we do? There are things for which we don't have enough numbers, e.g. points in the interval $(0,1)$
- Let S be a set of indexes, with members α . (Note that α is a *variable* in this usage.) Further, for each member of S let us assume we already have some set A_α . We define $\bigcup_\alpha A_\alpha$ and $\bigcap_\alpha A_\alpha$ exactly how?
- The special cases: when the index set is empty
- What are variables?
- What are families of sets?

PARTITIONING

- A *partition* of a set A is a family of sets A_α such that for any $\alpha \neq \beta$ we have $A_\alpha \cap A_\beta = \emptyset$ and $\bigcup_\alpha A_\alpha = A$. By definition, we never consider the empty set a part of any partition, so in the definition we may write “a family of *nonempty* sets”
- The partition can be finite (e.g. the sets of *even* and *odd* numbers partition the set of integers) or infinite
- Partitions are related to *equivalence relations*.
- There are two *trivial* partitions, when all elements are in the same set, and when all go in their own set

HOMEWORK 2

- Task 1: Define intersection and union for two sets, and show that union is the same as the notion of union used in ZFC
- Task 2: Show that union and intersection are commutative and associative, show that *both* distribute over the other
- Task 3: Define complementation relative to a fixed universe U , prove the De Morgan identities