Foundations of Mathematics, Lecture 10

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Proof theory

1. The goal: mechanical checking of proofs
2. We will manipulate proofs by elementary steps
3. Just one example: Gentzen 'natural deduction' and 'sequent calculus'
4. Every step fits the form $A_1, \ldots, A_n \vdash B_1, \ldots, B_k$ The sides are called *sequents*. WARNING: Left side interpreted conjunctively (',' means $\wedge$), right side interpreted disjunctively (',' means $\vee$) Natural deduction is the special case when $k = 1$
5. Why is this a good trick? Because implication $A \to B$ means $\neg A \vee B$ so when we move stuff to the other side it changes "sign" as in arithmetic. Just one axiom $p, r \vdash q, r$
6. Manipulation rules are written above and below a horizantal line

   e.g. $L \wedge$ rule: $$\frac{\Gamma, A \wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$$

Left:	Right:
$L\wedge$ rule: $\dfrac{\Gamma, A\wedge B \vdash \Delta}{\Gamma, A, B \vdash \Delta}$	$R\wedge$ rule: $\dfrac{\Gamma \vdash \Delta, A\wedge B}{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}$
$L\vee$ rule: $\dfrac{\Gamma, A\vee B \vdash \Delta}{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}$	$R\vee$ rule: $\dfrac{\Gamma \vdash \Delta, A\vee B}{\Gamma \vdash \Delta, A, B}$
$L\rightarrow$ rule: $\dfrac{\Gamma, A\rightarrow B \vdash \Delta}{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}$	$R\rightarrow$ rule: $\dfrac{\Gamma \vdash \Delta, A\rightarrow B}{\Gamma, A \vdash \Delta, B}$
$L\neg$rule: $\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma \vdash \Delta, A}$	$R\neg$rule: $\dfrac{\Gamma \vdash \Delta, \neg A}{\Gamma, A \vdash \Delta}$

Model theory

- We build *model structures* as sets endowed with the right kind of relations
- For example, a model of the natural numbers is a set M endowed with a succession relation $s \subset M \times M$ and satisfying the Peano axioms
- An *interpretation* is a mapping from formulas to a model
- Key notion: *elementary equivalence* holds between structures if they satisfy the same FOL sentences
- Lowenheim-Skolem theorem: if a theory has an infinite model, it has a countably infinite model 'downward L-S'
- If it has a countable model, it has a model at any cardinality 'upward L-S'

Cardinality

1. For finete sets, we just count the elements
2. We make some general observations in the finite case:
3. A subset cannot be larger, a proper subset must be smaller
4. We can make an injective mapping from smaller to larger-or-equal sets, and a surjective mapping from larger to smaller-or-equal
5. We can make a bijective mapping exactly when the two sets have the same size
6. We use some of these observations to *define* cardinality for infinite sets

Not all the above stays true in the infinite case

- A set can have the same cardinality as a proper subset: for example there are as many numbers as there are even numbers
- Also, as many even numbers as odd numbers
- Cardinality of the natural numbers is called $\aleph_0$
- Lots of things have this cardinality: integers, rationals, algebraic numbers, computable numbers, all finite subsets of the integers,...
- But not all subsets of the integers, 2^S is always strictly greater than S

The main theorems

- Cantor's Theorem: sets are strictly smaller than their powersets $|S| < |2^S|$
- Bernstein-Schröder Theorem: two injections make a bijection
- $|\mathbb{R}| > \aleph_0$
- Independence of Continuum Hypothesis (Gödel 1931 + Cohen 1963)

Homework, midterm

HW10.1: Prove that 'having the same cardinality' is an equivalence relation
HW10.2: Describe how to construct, by ruler and compass, an interval of length $\sqrt{x}$ given an interval of length 1 and and an interval of length x
HW10.3 Prove the Cantor-Bernstein-Schröder Theorem,
HW10.4-8 are CPZ 11.3; 11.5; 11.13; 11.21; and 11.31 respectively.

2nd Midterm starts at 12:15 on November 30. This will cover Chapters 1-11 of CPZ.