

FOUNDATIONS OF MATHEMATICS, LECTURE 8

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TRUTH

- There are two kinds of truth, syntactic and semantic
- We have \vdash 'yields' or 'derives' where $A \vdash B$ means B can be formally derived (proved) from A . For example, in most systems of logic $x = 3 \wedge y = x \vdash y = 3$, but we need a lot of machinery (called *proof theory*) to make this stick. This is pure syntax manipulation: you take formulas and produce new ones by mechanical operations
- We also have \models 'models' where $A \models B$ means that in any model where A is true B is also true. This is more meaningful, but requires *model theory* which spells out the relation between a theory (bunch of formulas) and a set with lots of structure that the formulas are about
- In well-crafted systems $A \vdash B$ implies $A \models B$

THE CONVERSE IS NOT TRUE!

- In many well-crafted systems (e.g. the first order formulation of Peano Arithmetic) there are statements which are semantically true e.g. $PA \models \text{Goodstein's Theorem}$, but *has no proof there*
- If it has no proof, how do we know it's true? Because in a stronger system (in this case, 2nd order arithmetic) we can prove it
- That the converse is not true for systems endowed with a bit of arithmetic is the celebrated Gödel Incompleteness Theorem
- Our interest here is with the less celebrated, but just as important, Gödel Completeness Theorem
- This says that every formula that is true in all structures is provable
- Wait, how can these both be true? The answer is that PA has more models in first-order axiomatization than in second-order

PROPOSITIONAL LOGIC

- Any statement that can be true or false (in either of the senses discussed above) is called a **proposition**. These come in two basic varieties: *a has property P* and *the relation R holds between some elements*. Examples of the first: *57 is prime*, of the second: $2 + 3 = 4$
- Things that are *not* propositions include imperatives *Go home!* and questions *Where is Johnny?*
- Declarative statements using variables are called **open propositions** 'x is prime'. These can get a truth value either by substitution '17 is prime' and '18 is prime' both have a truth value or by quantification (part of FOL, but not PL)
- ZFC Axiom 3 (comprehension) creates the connection between PL and set theory: for any open sentence $\phi(x, w_1, \dots, w_n)$ and any set A there exists a set B containing all and only those elements x of A for which $\phi(x, w_1, \dots, w_n)$ holds. 'elements of a set satisfying some proposition can be collected in a set'

BOOLEAN OPERATIONS

- Well, what are operations? Operations are like addition, multiplication, negation... How can we define operations?
- We don't need new machinery! *Binary* operations are **functions** with two variables. *Unary* operations are functions with one variable (minus, reciprocal, ...) *Nullary operations* are functions that don't depend on any variable, **constants**.
- A **structure** is a set S and some operations. For example groups have a nullary operation (the unit e), a unary operation ($^{-1}$), and a binary operation (multiplication) which satisfy some identities (group axioms). On occasion, we don't insist that an operation be everywhere defined.
- One set of operations that matters in PL are the Boolean \neg, \wedge, \vee
- These are 'truth functional' – only the truth of the operands matters for establishing the truth of the result

IMPLICATION IN PL

- 1 We define $P \rightarrow Q$ by $\neg P \vee Q$
- 2 This has interesting consequences, the most important being *ex falso quidlibet* 'everything follows from a false statement' or *if you start with a false premiss, you can derive any conclusion from it*
- 3 We also define truth-functional equivalence $p \equiv Q$ by $P \rightarrow Q \wedge Q \rightarrow P$
- 4 Remember sets satisfied the de Morgan identities $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$? In PL they are satisfied by $\neg(A \vee B) = \neg A \wedge \neg B$ and $\neg(A \wedge B) = \neg A \vee \neg B$
- 5 HW8: CPZ Chapter 2, exercises