Foundations of Mathematics, Lecture 5

András Kornai

BMETE91AM35 Fall 2020-21

Homework, midterm

- There is no homework to submit this week, because there will be a midterm next week
- Typical errors. VERY IMPORTANT: difference between $\{\}, \{\emptyset\}, \emptyset$; difference between (a, b) and $\{a, b\}$
- From now on homework deadline is Saturday 10PM CET. Two weeks from now, homeworks not submitted in LaTeXwill be penalized 25%
- Midterm can be handwritten! No penalty, but if you submit the same in LaTeXwithin 24 hours you will get extra points
- If you miss the midterm, and haven't submitted homework, you will get no signature in the course. If you fail the midterm, there will be an opportunity to make up, but only to those who otherwise participate

Functions as relations

- All functions are relations, but not all relations are functions!
- The big difference is that functions have *unique output*, a relation F will be called a function only if $aFb \wedge aFc \Rightarrow b = c$
- Definitions of domain, codomain, range, and composition are the same. We don't write 30° cos $\sqrt{3}/2$, we write $\cos 30^\circ = \sqrt{3}/2$
- CPZ devotes Chapter 10 to functions, we will cover this in class today, but the entire chapter is homework to read. Exercises similar to those in CPZ Ch 1, 9, and 10 will be on the midterm
- Composition of functions is just like composition of relations: if $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$
- Sometimes (often) more lax terminology is used, permitting functions to be defined only on a subset of their domain. For example, most people will talk about $\sqrt{}$ as an $\mathbb{R} \to \mathbb{R}$ function, even though its *natural domain* is only $\mathbb{R}_0^+$
- Other tricky point about $\sqrt{}$ is that output (depending on definition) is not unique

Main function types

- Functions are *defined* or *given* by their graphs, which are the set of (input, output) pairs. But we often think of functions as little machines that take some input and produce some output
- The input may be of different type than the output. Examples: distance travelled as a function of time; temperature as a function of space; force of gravity as a function of masses and distance, ...
- **Multivariate** functions don't depend on a single variable but several. For example, current is a function of both voltage and resistance (Ohm's Law)
- **Vector-valued** sometimes functions produce a k-tuple of values simultaneously. For example, at any given point in space gravity has both a magnitude and a direction (total of four numbers)
- These can happen at the same time: functions from n-tuples to k-tuples are often used

Variety of functions

- The central types are **numerical functions** from numbers to numbers. You will be seeing a lot of examples of *arithmetic* functions: domain $\mathbb{N}$ but range can be $\mathbb{R}$ or even $\mathbb{C}$
- Also very frequent are **real functions** with domain and range $\mathbb{R}$
- You will love **complex functions** with domain and range $\mathbb{C}$
- **Functionals** are functions whose domain are functions, and range is typically $\mathbb{R}$ or $\mathbb{C}$
- **Operators** are functions from functions to functions
- All of these are heavily used in physics/engineering
- But there is more! Not all functions involve numbers, for example the truth function maps formulas onto the set {true, false }
- We will also have a lot to say about **operations** in algebra

Main properties of functions

1. Injective: different x-es map on different y-s:
   $f(x) = f(y) \Rightarrow x = y$
2. Surjective: codomain = range (codomain $\supset$ range is true by definition)
3. Bijective: both injective and surjective
4. **Theorem:** a function f is *invertible* $\Leftrightarrow$ it is *bijective*
5. **Proof:** We need to prove both $\Rightarrow$ and $\Leftarrow$. For $\Rightarrow$ we need to *verify* that the bijective properties follow from invertibility. For $\Leftarrow$ we will construct the inverse of a bijective function.
6. ($\Rightarrow$) What do we suppose? What do we need to prove?
7. ($\Leftarrow$) What do we suppose? What do we need to prove?
8. Discussion of CPZ Ex 10.18