Foundations of Mathematics, Lecture 4

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BMETE91AM35 Fall 2020-21

Administrative matters

- Missing students (in Neptun, but no homeworks/attendance) QDKW3E; PX7CNT; B6OPBR; BX0DYA; K5JES7; N3FJLA; OPYMAH; U1G6A8;
- Some people missing one homework *please do them!*
- All material available at https://kornai.com/2020/FoundationsOfMathematics

We have the foundations, let's build the house!

- We actually only have half of the foundations (set theory), we will still need *mathematical logic* to make this airtight, but we can start building
- **Relations** A (binary) relation R with domain X and codomain Y is a subset of $X \times Y$ such that $\forall a \in X \exists b \in Y : (a, b) \in R$, and conversely, any such subset is a relation with domain X and codomain Y
- The most important relations are **equality** $=$; **similarity** $\sim$; and **ordering** $>$ or $\geq$
- Within set theory, we have only two relations treated as primitive: $\in$ (element of) and $=$ (equal)
- These two are linked by ZFC1
- Difference between codomain and range!

Relations

- Language offers many examples *x causes y; x has y; x is y ...*
- In fact logical syntax often treats all verbs as a relation between subject and object: *x loves y* is written as xLy or L(x,y)
- More complex verbs may require ternary relations (*Give(x,y,z); Rent(x,y,z,t,p)*)
- Key idea: an ordered triple can be treated with *left association* as ((a,b),c) or with *right association* as (a,(b,c))
- In Kuratowski encoding:
  $\{\{\{a\}\{ab\}\}\{\{\{a\}\{ab\}\}\{\{a\}\{ab\}\}\{c\}\}\}$ or
  $\{\{a\}\{\{a\}\{\{b\}\{bc\}\}\}\}$
- HW4.1 State the definition of what ot means for the style of encodings to be isomorpic, and prove that they are in fact isomorphic

Major properties of binary relations

- For any relation $R \subset (X \times Y)$ we can produce the *reversal* R^T and the *complement* R^C of R
- These are not the same! For equality $=^T$ is $=$ but $=^C$ is $\neq$
- HW4.2 Write formulas for R^T and R^C!
- A relation is **reflexive** iff $\forall a \in X : aRa$
- A relation is **irreflexive** iff $\nexists a \in X : aRa$
- A relation is **symmetric** iff $\forall ab : aRb \Rightarrow bRa$
- A relation is **antisymmetric** iff $\forall ab : aRb \wedge bRa \Rightarrow a = b$
- A relation is **transitive** iff $\forall abc : aRb \wedge bRc \Rightarrow aRc$
- Relations are sets: we can do union, intersection, complementation (relative to universe $X \times Y$)
- Composition of relations can also be defined provided types match: $R \subset X \times Y, S \subset Y \times Z$. We say $a(S \circ R)c \Leftrightarrow \exists b : aRb \wedge bSc$

Major types of binary relations

1. Relations that enjoy the reflexive, symmetrical, and transitive properties are called **equivalence relations**
2. Relations that are reflexive, antisymmetric, and transitive are called **ordering relations**
3. Equivalence relations are covered in Chapter 9 of CPZ
4. They are closely related to partitions: every e.r. corresponds to a partition
5. We discuss CPZ 9.1-3 in class
6. HW4.3 is CPZ9.4
7. Divisibility (CPZ9.6). $a|b \Leftrightarrow b/a \in \mathbb{N}$
8. We do this now for natural numbers $\{1, 2, \ldots\}$, but it extends smoothly to integers $\mathbb{Z}$
9. HW4.4– is CPZ9.8,10,12,14,16,18,20,22