About the 0th test

- Two people got both problems right, some got both wrong, most people in between
- In Problem 1, we will begin with the standard (ZFC) axioms, and see how these lead to the conclusion
- In Problem 2 (perhaps next class), we will begin with the solution, and go back towards axioms
- We will start building LaTeXskills: people will have to typeset ZFC
- We will start using Chartrand-Polimeni-Zhang

Zermelo-Fraenkel Axioms

1. Axiom of Extensionality: If X and Y have the same elements, then $X = Y$.

$$\forall u\,(u \in X \equiv u \in Y) \Rightarrow X = Y. \tag{1}$$

2. Axiom of the Unordered Pair: For any a and b there exists a set $\{a,\ b\}$ that contains exactly a and b. (also called Axiom of Pairing)

$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \equiv (x = a \vee x = b)). \tag{2}$$

3. Axiom of Subsets: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \varphi(u,\ p)\}$ that contains all those $u \in X$ that have the property φ. (also called Axiom of Separation or Axiom of Comprehension)

$$\forall X\,\forall p\,\exists Y\,\forall u\,(u \in Y \equiv (u \in X \wedge \varphi(u,\ p))). \tag{3}$$

4. Axiom of the Sum Set: For any X there exists a set $Y = \bigcup X$, the union of all elements of X. (also called Axiom of Union)

$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv \exists z\,(z \in X \wedge u \in z)). \tag{4}$$

5. Axiom of the Power Set: For any X there exists a set $Y = P(X)$, the set of all subsets of X.

$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv u \subseteq X). \tag{5}$$

6. Axiom of Infinity: There exists an infinite set.

$$\exists S\,[\emptyset \in S \wedge (\forall x \in S)\,[x \bigcup \{x\} \in S]]. \tag{6}$$

7. Axiom of Replacement: If F is a function, then for any X there exists a set $Y = F[X] = \{F(x) : x \in X\}$.

$$\begin{aligned} &\forall x\,\forall y\,\forall z\,[\varphi(x,\ y,\ p) \wedge \varphi(x,\ z,\ p) \Rightarrow y = z] \\ &\Rightarrow \forall X\,\exists Y\,\forall y\,[y \in Y \equiv (\exists x \in X)\,\varphi(x,\ y,\ p)]. \end{aligned} \tag{7}$$

8. Axiom of Foundation: Every nonempty set has an $\in$-minimal element. (also called Axiom of Regularity)

$$\forall S\,[S \neq \emptyset \Rightarrow (\exists x \in S)\,S \bigcap x = \emptyset]. \tag{8}$$

9. Axiom of Choice: Every family of nonempty sets has a choice function.

$$\forall x \in a\,\exists A(x,\ y) \Rightarrow \exists y\,\forall x \in a\,A(x,\ y(x)). \tag{9}$$

LATEX BASICS

- You need to install latex on your laptop. Visit https://www.latex-project.org/get/ and choose one (TexLive for Linux, MacTex for Mac, or MikTex for Windows), download, and try it out!
- Some of these come with a full graphical interface, but you can use just an ordinary plain text editor (emacs, vi, vim, nano, etc) in a terminal window
- There is a write-compile-debug cycle: start with file.tex, produce file.pdf, test, repeat (a bit more complex if a bibliography is involved)
- Your first homework will be to write ZFC in LaTeX
- Yes, you can do this in your own language, not just English!
- Download `zfcskel.tex` from the course webpage and edit that. Start with English even if you want to learn LaTeXfor some other script or language

The basic structure of your document

```
\documentclass{article}
\usepackage{colortbl}
\author{YOUR NAME}
\title{The ZFC axioms of set theory}
\date{}
\begin{document}
\maketitle

\begin{enumerate}

\item {\color{green} Extensionality} If $X$ and $Y$ ...
\begin{equation}
\forall u ...
\end{equation}

\item {\color{green} Unordered Pair} For any $a$ and $b$
\begin{equation}
\forall a \forall b
\end{equation}
...

\item {\color{green} Choice}  Every family of nonempty sets...
\begin{equation}
\forall x \in a \exists A(x,y) \Rightarrow ...
\end{equation}
\end{enumerate}

\end{document}
```

Reasons why all mathematicians use LaTeX

1. Typesetting formulas/equations is *hard*, LaTeXmakes is easy
2. Too many symbols are used to fit on any keyboard. It is much better to learn a few names, like `\forall` for $\forall$, `\exists` for $\exists$, ...
3. Clear separation of format and content
4. Clear separation of "object language" and "metalanguage"
5. Can do all kinds of stuff, but runs everywhere
6. Can do much better typesetting than Word or really anything else
7. Well integrated with rest of the ecosystem (e.g. MathJax for webpages)

What is real?

- This is perhaps the deepest question in philosophy
- Are triangles real? In what way do they exist? Can you find them in nature?
- Do the rules of soccer exist? In what way? Can you find them in nature?
- Let's agree that *things* like trees or houses exist! [To see that this is by no means a fully accepted position, take a look at https://en.wikipedia.org/wiki/Phenomenalism]
- The rules of soccer are like houses: they are man-made (can't be found in nature)
- Math is like soccer: it is a game. It's fun, and just like soccer, it is useful (homework: think of reasons why soccer is useful, or whether it's useful at all)
- Mathematicians agree that sets exist

In what way do sets exist?

- They exist because we accept the ZF(C) axioms
- We could permit other existing objects, in particular we could permit numbers, triangles, functions, equations, ...
- But there is no need! Once you acknowledge the existence of sets, all these other things come for free
- How much you need to acknowledge? Let's start with the empty set.
- Can we even *prove* the existence of $\emptyset$?
- The easy way out: let's make it an axiom! How do you formulate this axiom?

The empty set

- $\exists x \forall y \neg (y \in x)$
- Can there be more than one? Is this a stupid question?
- No, most people would agree that the set of triangular circles and the set of pink elephants are different, even though they are both empty!
- Well, in set theory there can only be one unique object. Why?
- Suppose u and v are both empty sets. What does Axiom 1 tell us?
- So if there is one, it's unique. But how to guarantee existence?
- Let's look at the axioms again!

More on the empty set

- $\emptyset$ must be some x that satisfies $\forall y \neg(y \in x)$
- But this $\forall y \neg(y \in x)$ is a *property*, and we have Axiom 3 (the subset axiom) that if ϕ is a property with parameter p than for any X and p there exists a set Y that contains exactly those elements of X that satisfy ϕ
- Now if we permit just one set Y, which need not be empty, we can derive the existence of $\emptyset$ by using the above property. Thus, Axiom 3 offers a means of *constructing* $\emptyset$ from any set Y
- But what if there are no sets whatsoever? Well, in that case there doesn't remain any common basis for discussing anything with mathematicians! (A) go home, and study some other subject. (B) admit that there exist *some* sets
- There have been very serious attempts to demonstrate that ZFC has contradictions, there are no sets fulfilling the axims. None of these have succeeded so far, so we still believe ZFC set theory is a reasonable basis for all of mathematics.

One last word on the empty set

- There are other set theories, as many as there are axiom systems. In particular ZF without C is different from ZF with C. Other well-studied systems include von Neumann-Gödel-Bernays (NGB), Morse-Kelley (MK), and Kripke-Patek – we may discuss these a bit. There are also interesting systems such as Aczel's, which replaces Axiom 8 of ZFC by an Axiom of Anti-Foundation
- Look at the ZFC axioms more carefully. Do you see some other means to get to $\emptyset$? Why yes, there is Axiom 6. What does it do for us?
- First of all, it guarantees the existence of a particular infinite set
- But it does something more! It already guarantees the existence of $\emptyset$
- Depending on your version of set theory, you may have to (A) have an axiom that guarantees its existence or (B) derive it from other axioms
- There is also (C), trying to get by without $\emptyset$. This is doable, but

The first problem from the 0th test

Three subsets A, B, and C of $\{1,2,3,4,5\}$ have the same cardinality. Furthermore

A 1 belongs to A and B but not to C

B 2 belongs to A and C but not to B

C 3 belongs to A and exactly one of B and C

D 4 belongs to an even number of A, B, and C

E 5 belongs to an odd number of A, B, and C

F The sums of the elements in two of the sets A, B, and C differ by 1

	1	2	3	4	5
A	$+$	$+$	$+$	$0/2$	$1/3$
B	$+$	$-$	$\vee$	$0/2$	$1/3$
C	$-$	$+$	$\wedge$	$0/2$	$1/3$

Applying the other conditions

- So far, we know that A has at least 3 elements. Can it have 5?
- No, because in that case B and C must also have 5 elements, so they all would be the same set by Axiom 1.
- Can they all have four? No, because that would require 4 and 5 to appear in both B and C, and that would still not be enought, because 3 appears in only one of these!
- So we are done with A, and we now know that 5 cannot appear in A, so condition *E* means we must have 1 occurrence of 5 (either in B or in C, not both)

	1	2	3	4	5
A	$+$	$+$	$+$	$-$	$-$
B	$+$	$-$	$\vee$	0/2	$\wedge$
C	$-$	$+$	$\wedge$	0/2	$\vee$

Applying the other conditions

Since B will have one of 3 and 5, and C will have the other, we must have 4 both in B and C. This leaves us two possibilities:

	1	2	3	4	5
A	+	+	+	−	−
B	+	−	+	+	−
C	−	+	−	+	+

	1	2	3	4	5
A	+	+	+	−	−
B	+	−	−	+	+
C	−	+	+	+	−

So the sums of the numbers in the sets will be

	1	2	3	4	5	
A	+	+	+	−	−	6
B	+	−	+	+	−	8
C	−	+	−	+	+	10

	1	2	3	4	5	
A	+	+	+	−	−	6
B	+	−	−	+	+	10
C	−	+	+	+	−	9

Homework

- Produce a .tex and .pdf version of the ZFC axioms given on Slide 2 above. You can start with `zfcskel.tex` from the course webpage
- Solve all 9 problems in Section 1.1 of Chartrand-Polimeni-Zhang (pp 17–18). You can submit your results handwritten (just send me a picture) but you get extra points for doing the problems in LaTeX.
- Later problem sets will all be due in LaTeX, this is the last week you can submit handwritten stuff