Foundations of Mathematics, Lecture 12

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Cardinality

1. For finete sets, we just count the elements
2. We make some general observations in the finite case:
3. A subset cannot be larger, a proper subset must be smaller
4. We can make an injective mapping from smaller to larger-or-equal sets, and a surjective mapping from larger to smaller-or-equal
5. We can make a bijective mapping exactly when the two sets have the same size
6. We use some of these observations to *define* cardinality for infinite sets

Not all the above stays true in the infinite case

- A set can have the same cardinality as a proper subset: for example there are as many numbers as there are even numbers
- Also, as many even numbers as odd numbers
- Cardinality of the natural numbers is called $\aleph_0$
- Lots of things have this cardinality: integers, rationals, algebraic numbers, computable numbers, all finite subsets of the inegers,...
- But not all subsets of the integers, 2^S is always strictly greater than S

The main theorems

- Cantor's Theorem: sets are strictly smaller than their powersets $|S| < |2^S|$
- Bernstein-Schröder Theorem: two injections make a bijection
- $|\mathbb{R}| > \aleph_0$
- Independence of Continuum Hypothesis (Gödel 1931 + Cohen 1963)

Homework, midterm

HW12.1: Prove that 'having the same cardinality' is an equivalence relation
HW12.2: Describe how to construct, by ruler and compass, an interval of length $\sqrt{x}$ given an interval of length 1 and and an interval of length x
HW12.3 Prove the Cantor-Bernstein-Schröder Theorem,
HW12.4-8 are CPZ 11.3; 11.5; 11.13; 11.21; and 11.31 respectively.

2nd Midterm starts at 12:15 on December 1st. Chapter 11 of CPZ *will appear* on the midterm, but Chapter 12, which we will discuss afterwards, will not.