About the 0th test, 2nd problem

- The task was to (a) *compute* $\cos(20^\circ)\cos(40^\circ)\cos(80^\circ)$ and (b) *prove* that this is exactly $1/8$.
- Computing it to four-five decimals and obtain 0.12500 is trivial
- Many people used a trick, multiplying by, and also dividing by, $\sin(20^\circ)$ and relying on formulas of multiple angles
- This is fine, as long as we can prove these formulas as lemmas
- We will develop a very different proof, which doesn't rely on the axiomatization of geometry that much

Proof outline

- We will prove the well-known addition formulas $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- Wou may have learned the geometric proof for these in high scool: for detailed explanation of these there are videos e.g. on Khan Academy
- These actually require proving lots of lemmas which high-school geometry doesn't really teach rigorously
- We will return to this part of the proof when you have already studied complex numbers, matrix multiplication and isomorphy
- Once these pieces of algebra are at hand, the addition formulas will be trivial, but the pieces (including the Euler and de Moivre formulas will be super-useful for many other things as well!
- We will need other pieces of algebra, in particular, you will also learn about polynomials and structures before we can complete the proof. But these things are again super-useful on their own

Discussion of LaTeXissues

- Anybody has installation issues? We are here to help
- Anybody had serious problems with the ZFC homework? We are here to help
- This is not a course in LaTeX, there will be no more specifically LaTeXhomeworks
- But if you are still missing the ZFC homework, fix this soonest, or it will affect your grade
- All subsequent homeworks are due in LaTeX
- Homeworks will be evaluated on content, not formatting, but correct formatting is expected for the rest of your life. You will be grateful that we forced this on you!
- Also, teachers in other courses will love you for it

Set operations

1. Union, intersection, but no complementation in general – why not?
2. Subscripting, indexing
3. Partitioning
4. Cartesian (also called direct) product
5. Ordered pairs by sets
6. Alternatives to the standard definition
7. Back to direct products

Boolean operations

- When there is a base set (such as the set of integers) everything works!
- We get the de Morgan identities: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Complementation is an involution $\overline{\overline{A}} = A$
- $\cup$ and $\cap$ are commutative, associative, idempotent
- Two kinds of distributive laws
- There is a 0 (the empty set) and a 1 (the universe)
- Will be generalized to *lattices* later on

Trouble with negation

- There is no "set of all sets"
- Why not? Because it would not be well founded (doable with AFA, but not with ZFC)
- Because it would give rise to Russel's Paradox: by Comprehension we could form the set of all sets that don't contain themselves!
- Paradox can be avoided by (a) positive comprehension (b) type theory
- Most math is done with (b) – types are built into ZFC
- Strongly related to type checking in programming languages
- You can have the *class* of all sets, and other classes (NGB set theory)

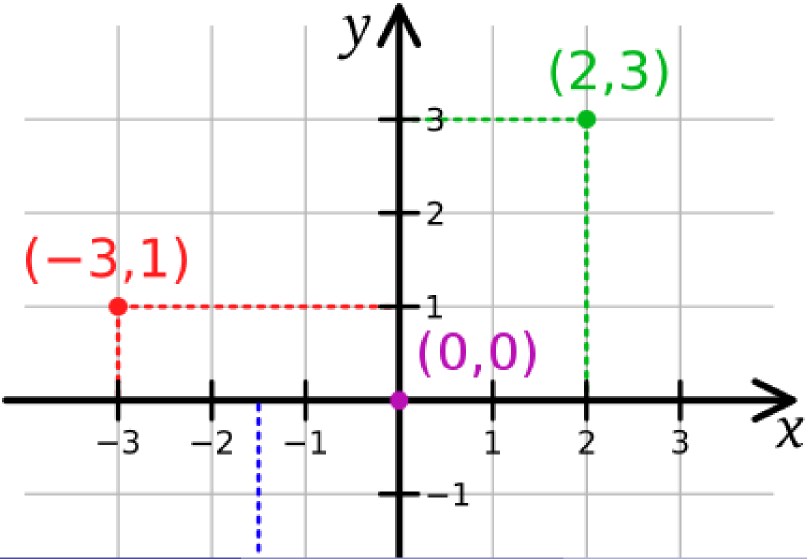
Subscripting, indexing

- The basic idea: instead of $A, B, C, \ldots, Z$ and running out of letters after 26, let's do $A_1, A_2, \ldots, A_{777}$ because we never run out of numbers
- But what if we do? There are things for which we don't have enough numbers, e.g. points in the interval (0,1)
- Let S be a set of indexes, with members α. (Note that α is a *variable* in this usage.) Further, for each member of S let us assume we already have some set A_α. We define $\bigcup_\alpha \mathrm{A}_\alpha$ and $\bigcap_\alpha A_\alpha$ exactly how?
- The special cases: when the index set is empty
- What are variables?
- What are families of sets?

Partitioning

- A *partition* of a set A is a family of sets A_α such that for any $\alpha \neq \beta$ we have $A_\alpha \cap A_\beta = \emptyset$ and $\bigcup_\alpha \mathrm{A}_\alpha = A$. By definition, we never consider the empty set a part of any partitition, so in the definition we may write "a family of *nonempty* sets"
- The partition can be finite (e.g. the sets of *even* and *odd* numbers partition the set of integers) or infinite
- Partitions are related to *equivalence relations*.
- There are two *trivial* partitions, when all elements are in the same set, and when all go in their own set

Cartesian products: the idea



Cartesian products: implementing the idea

- We need *ordered pairs*, where $(1, 2) \neq (2, 1)$
- One approach (Haussdorf 1914): number the coordinates: define (x, y) by $\{\{x, 1\}, \{y, 2\}\}$. This gets trickier when x, y are numbers, and at any rate it presupposes the idea of a number
- The standard approach (Kuratowski 1921) $(x, y) \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$
- Variants are possible, don't bring much novelty
- Now we can build the cartesian or *direct* product $A \times B$ of two sets A and B as $\{(a, b) | a \in A, b \in B\}$
- How do we extend this to n-tuples?
- This will lead us to a bunch of useful definitions, for *relation, function, operation* etc.